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ADVERTISEMENT.

The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

PHILOSOPHICAL TRANSACT

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Propagation of Tremors over the Surface of an Elastic Solid.*

By HORACE LAMB, *F.R.S.*

Received June 11,—Read June 11,—Revised October 28, 1903.

INTRODUCTION.

1. THIS paper treats of the propagation of vibrations over the surface of a "semi-infinite" isotropic elastic solid, *i.e.*, a solid bounded only by a plane. For purposes of description this plane may be conceived as horizontal, and the solid as lying below it, although gravity is not specially taken into account.*

The vibrations are supposed due to an arbitrary application of force at a point. In the problem most fully discussed this force consists of an impulse applied vertically to the surface; but some other cases, including that of an internal source of disturbance, are also (more briefly) considered. Owing to the complexity of the problem, it has been thought best to concentrate attention on the vibrations as they manifest themselves at the free surface. The modifications which the latter introduces into the character of the waves propagated into the interior of the solid are accordingly not examined minutely.

The investigation may perhaps claim some interest on theoretical grounds, and also in relation to the phenomena of earthquakes. Writers on seismology have naturally endeavoured from time to time to interpret the phenomena, at all events in their broader features, by the light of elastic theory. Most of these attempts have been based on the laws of wave-propagation in an unlimited medium, as developed by GREEN and STOKES; but Lord RAYLEIGH's discovery† of a special type of surface-waves has made it evident that the influence of the free surface in modifying the character of the vibrations is more definite and more serious than had been suspected. The present memoir seeks to take a further step in the adaptation of theory to actual conditions, by investigating cases of *forced* waves, and by abandoning (ultimately) the restriction to simple-harmonic vibrations. Although the circumstances of actual earthquakes must differ greatly from the highly idealized state of things which we are obliged to assume as a basis of calculation, it is hoped that the solution of the problems here considered may not be altogether irrelevant.

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* Professor BROMWICH has shown ('Proc. Lond. Math. Soc.,' vol. 30, p. 98 (1898)) that in such problems as are here considered the effect of gravity is, from a practical point of view, unimportant.

† 'Proc. Lond. Math. Soc.,' vol. 17, p. 4 (1885); 'Scientific Papers,' vol. 2, p. 441.

It is found that the surface disturbance produced by a single impulse of short duration may be analysed roughly into two parts, which we may distinguish as the "minor tremor" and the "main shock," respectively. The minor tremor sets in at any place, with some abruptness, after an interval equal to the time which a wave of longitudinal displacement would take to traverse the distance from the source. Except for certain marked features at the inception, and again (to a lesser extent) at an epoch corresponding to that of direct arrival of transversal waves, it may be described, in general terms, as consisting of a long undulation leading up to the main shock, and dying out gradually after this has passed. Its time-scale is more and more protracted, and its amplitude is more and more diminished, the greater the distance from the source. The main shock, on the other hand, is propagated as a solitary wave (with one maximum and one minimum, in both the horizontal and vertical displacements); its time-scale is constant; and its amplitude diminishes only in accordance with the usual law of annular divergence, so that its total energy is maintained undiminished. Its velocity is that of free Rayleigh waves, and is accordingly somewhat less than that of waves of transversal displacement in an unlimited medium.*

The paper includes a number of subsidiary results. It is convenient to begin by attacking the problems in their two-dimensional form, calculating (for instance) the effect of a pressure applied uniformly along a *line* of the surface. The interpretation of the results is then comparatively simple, and it is found that a good deal of the analysis can be utilized afterwards for the three-dimensional cases. Again, the investigation of the effects of a simple-harmonic source of disturbance is a natural preliminary to that of a source varying according to an arbitrary law.

Incidentally, new solutions are given of the well-known problems where a periodic force acts transversely to a line, or at a point, in an unlimited solid. These serve, to some extent, as tests of the analytical method, which presents some features of intricacy.

2. A few preliminary formulæ and conventions as to notation may be put in evidence at the outset, for convenience of reference.

The usual notation of BESSEL's Functions "of the first kind" is naturally adhered to; thus we write:

$$J_0(\zeta) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos(\zeta\cos\omega)\,d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

---

* Compare the concluding passage of Lord RAYLEIGH's paper:

"It is not improbable that the surface-waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance."

The calculations indicate that the preponderance is much greater than would be inferred from a mere comparison of the ordinary laws of two-dimensional and three-dimensional divergence.

By a known theorem we have also

$$J_0(\zeta) = \frac{2}{\pi}\int_0^\infty \sin(\zeta \cosh u)\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

provided ζ be real and positive. For our present purpose it is convenient to follow H. **Weber**[*] in adopting as the standard function "of the second kind"

$$K_0(\zeta) = \frac{2}{\pi}\int_0^\infty \cos(\zeta \cosh u)\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

It is further necessary to have a special symbol for that combination of the two functions (2) and (3) which is appropriate to the representation of a diverging wave-system; we write, after Lord **Rayleigh**,[†]

$$D_0(\zeta) = \frac{2}{\pi}\int_0^\infty e^{-i\zeta \cosh u}\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

so that

$$D_0(\zeta) = K_0(\zeta) - iJ_0(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We shall also write, in accordance with the usual conventions,

$$J_1(\zeta) = -J'_0(\zeta), \quad K_1(\zeta) = -K'_0(\zeta), \quad D_1(\zeta) = -D'_0(\zeta) \quad . \quad . \quad (6).$$

For large values of ζ we have the asymptotic expansion

$$D_0(\zeta) = \sqrt{\frac{2}{\pi\zeta}} \cdot e^{-i(\zeta+\frac{1}{4}\pi)} \left\{1 - \frac{1^2}{1!\,(8i\zeta)} + \frac{1^2 \cdot 3^2}{2!\,(8i\zeta)^2} - \ldots\right\} \quad . \quad . \quad . \quad (7).$$

In the two-dimensional problems of this paper we shall have to deal with a number of solutions of the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + h^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

constructed from the type

$$\phi = Ae^{-\alpha y}\, e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

where ξ is real, and

$$\alpha = \sqrt{(\xi^2 - h^2)}, \quad \text{or} = i\sqrt{(h^2 - \xi^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

---

[*] 'Part. Diff.-Gleichungen d. math. Physik,' Brunswick, 1899–1901, vol. 1, p. 175. **Heine** ('Kugelfunctionen,' Berlin, 1878–1881, vol. 1, p. 185) omits the factor $2/\pi$. In terms of the more usual notation,

$$K_0 = \frac{2}{\pi}\{-Y_0 + (\log 2 - \gamma)J_0\},$$

where γ is **Euler**'s constant. The function $\frac{1}{2}\pi K_0$ has been tabulated (see J. H. **Michell**, 'Phil. Mag.,' Jan., 1898).

[†] 'Phil. Mag.,' vol. 43, p. 259 (1897); 'Scientific Papers,' vol. 4, p. 283. I have introduced the factor $2/\pi$, and reversed the sign.

according as $\xi^2 \gtrless h^2$, the radicals being taken positively. In particular, we shall meet with the solution

$$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha y}\, e^{i\xi x}\, d\xi}{\alpha} = \frac{2}{\pi}\int_{0}^{\infty} \frac{e^{-\alpha y}\cos \xi x\, d\xi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11);$$

and it is important to recognize that this is identical with $D_0(hr)$, where $r = \sqrt{(x^2 + y^2)}$. To see this, we remark that ϕ, as given by (11), is an even function of x, and that for $x = 0$ it assumes the form

$$\phi = \frac{2}{\pi}\int_{0}^{\infty} \frac{e^{-\alpha y}\, d\xi}{\alpha} = \frac{2}{\pi}\int_{0}^{\infty} \frac{e^{-i\sqrt{(h^2+\eta^2)}y}\, d\eta}{\sqrt{(h^2 + \eta^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

by the method of contour-integration.* This is obviously equal to $D_0(hy)$. Again, the mean value of any function ϕ which satisfies (8), taken round the circumference of a circle of radius r which does not enclose any singularities, is known to be equal to $J_0(kr).\phi_0$, where ϕ_0 is the value at the centre.† We can therefore adapt an argument of THOMSON and TAIT‡ to show that a solution of (8) which has no singularities in the region $y > 0$, and is symmetrical with respect to the axis of y, is determined by its values at points of this axis. We have, accordingly,

$$D_0(hr) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha y}\, e^{i\xi x}\, d\xi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Again, in some three-dimensional problems where there is symmetry about the axis of z, we have to do with solutions of

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + h^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

based on the type

$$\phi = Ae^{-\alpha z}\, J_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where $\varpi = \sqrt{(x^2 + y^2)}$, and α is defined as in (10). In particular, we have the solution

$$\phi = \int_{0}^{\infty} \frac{e^{-\alpha z}\, J_0(\xi\varpi)\, \xi\, d\xi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

which (again) reduces to a known function. At points on the axis of symmetry $(\varpi = 0)$ it takes the form

$$\phi = \int_{0}^{\infty} \frac{e^{-\alpha z}\, \xi\, d\xi}{\alpha} = \int_{ih}^{\infty} e^{-\alpha z}\, d\alpha = \frac{e^{-ihz}}{z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

---

[*] If we equate severally the real and imaginary parts in the second and third members of (12), we reproduce known results.

[†] H. WEBER, 'Math. Ann.,' vol. 1 (1868).

[‡] 'Natural Philosophy,' § 498.

Since the mean value of a function ϕ which satisfies (14), taken over the surface of a sphere of radius r not enclosing any singularities, is equal to

$$\frac{\sin hr}{hr} \cdot \phi_0,$$

where ϕ_0 is the value at the centre,[*] the argument already borrowed from THOMSON and TAIT enables us to assert that

$$\frac{e^{-ihr}}{r} = \int_0^\infty \frac{e^{-az} \mathrm{J}_0(\xi\varpi)\, \xi \, d\xi}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),\dagger$$

where

$$r = \sqrt{(\varpi^2 + z^2)} = \sqrt{(x^2 + y^2 + z^2)}.$$

Finally, we shall require FOURIER's Theorem in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(\lambda)\, e^{i\xi(x-\lambda)}\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),\ddagger$$

and the analogous formula

$$f(\varpi) = \int_0^\infty \mathrm{J}_0(\xi\varpi)\, \xi \, d\xi \int_0^\infty f(\lambda)\, \mathrm{J}_0(\xi\lambda)\, \lambda \, d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).\S$$

As particular cases, if in (19) we have $f(x) = 1$ for $x^2 < a^2$, and $= 0$ for $x^2 > a^2$, then

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \xi a}{\xi} e^{i\xi x} \, d\xi = \frac{2}{\pi} \int_0^\infty \frac{\sin \xi a}{\xi} \cos \xi x \, d\xi \quad . \quad . \quad . \quad . \quad (21);$$

and, if in (20) $f(\varpi) = 1$ for $\varpi < a$, and $= 0$ for $\varpi > a$, then

$$f(\varpi) = a \int_0^\infty \mathrm{J}_0(\xi\varpi)\, \mathrm{J}_1(\xi a)\, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

These are of course well-known results.[‖]

---

[*] H. WEBER, 'Crelle,' vol. 69 (1868).

[†] If in (18) we put $z = 0$, and then equate separately the real and imaginary parts, we deduce

$$\int_0^\infty \mathrm{J}_0(\zeta \cosh u) \cosh u \, du = \frac{\cos \zeta}{\zeta},$$

$$\int_0^\infty \mathrm{J}_0(\zeta \sin u) \sin u \, du = \frac{\sin \zeta}{\zeta}.$$

These are known results. *Cf.* RAYLEIGH, 'Scientific Papers,' vol. 3, pp. 46, 98 (1888); HOBSON, 'Proc. Lond. Math. Soc.,' vol. 25, p. 71 (1893); and SONINE, 'Math. Ann.,' vol. 16).

[‡] H. WEBER, 'Part. Diff.-Gl. etc.,' vol. 2, p. 190. Since λ occurs here and in (20) only as an intermediate variable, no confusion is likely to be caused by its subsequent use to denote an elastic constant.

[§] H. WEBER, 'Part. Diff.-Gl. etc.,' vol. 1, p. 193.

[‖] It may be noticed that if in (20) we put $f(\varpi) = e^{-ih\varpi}/\varpi$, we reproduce formulæ given in the foot-note † above.

PART I.

Two-Dimensional Problems.

3. The equations of motion of an isotropic elastic solid in two dimensions (x, y) are

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u, \quad \rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial y} + \mu \nabla^2 v \quad . \quad . \quad (23),$$

where u, v are the component displacements, ρ is the density, λ, μ are the elastic constants of Lamé, and

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

These equations are satisfied by

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),^*$$

provided

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2 \phi, \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

In the case of simple-harmonic motion, the time-factor being e^{ipt}, the latter equations take the forms

$$(\nabla^2 + h^2)\phi = 0, \quad (\nabla^2 + k^2)\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

where

$$h^2 = \frac{p^2 \rho}{\lambda + 2\mu} = p^2 a^2, \quad k^2 = \frac{p^2 \rho}{\mu} = p^2 b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

the symbols a, b denoting (as generally in this paper) the wave-*slownesses*,† *i.e.*, the reciprocals of the wave-velocities, corresponding to the irrotational and equivoluminal types of disturbance respectively.

The formulæ (25) now give, for the component stresses,

$$\left.\begin{aligned}
\frac{p_{xx}}{\mu} &= \frac{\lambda}{\mu}\Delta + 2\frac{\partial u}{\partial x} = -k^2\phi - 2\frac{\partial^2 \phi}{\partial y^2} + 2\frac{\partial^2 \psi}{\partial x\,\partial y} \\
\frac{p_{xy}}{\mu} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial^2 \phi}{\partial x\,\partial y} - k^2\psi - 2\frac{\partial^2 \psi}{\partial x^2} \\
\frac{p_{yy}}{\mu} &= \frac{\lambda}{\mu}\Delta + 2\frac{\partial v}{\partial y} = -k^2\phi - 2\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 \psi}{\partial x\,\partial y}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (29).$$

---

* Green, 'Camb. Trans.,' vol. 6 (1838); 'Math. Papers,' p. 261.

† The introduction of special symbols for wave-slownesses rather than for wave-velocities is prompted by analytical considerations. The term "wave-slowness" is accredited in Optics by Sir W. R. Hamilton.

In the applications which we have in view, the vibrations of the solid are supposed due to prescribed forces acting at or near the plane $y = 0$. We therefore assume as a typical solution of (27), applicable to the region $y > 0$,

$$\phi = \mathrm{A}e^{-\alpha y}e^{i\xi x}, \quad \psi = \mathrm{B}e^{-\beta y}e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

where ξ is real, and α, β are the positive real, or positive imaginary,* quantities determined by

$$\alpha^2 = \xi^2 - h^2, \quad \beta^2 = \xi^2 - k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

For the region $y < 0$, the corresponding assumption would be

$$\phi = \mathrm{A}'e^{\alpha y}e^{i\xi x}, \quad \psi = \mathrm{B}'e^{\beta y}e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

The time-factor is here (and often in the sequel) temporarily omitted.

The expressions (30), when substituted in (25) and (29), give for the displacements and stresses at the plane $y = 0$

$$u_0 = (i\xi\mathrm{A} - \beta\mathrm{B})\,e^{i\xi x}, \quad v_0 = (-\,\alpha\mathrm{A} - i\xi\mathrm{B})\,e^{i\xi x} \quad . \quad . \quad . \quad . \quad (33),$$

and

$$\left.\begin{aligned} [\,p_{xy}]_0 &= \mu\,\{-\,2i\xi\alpha\mathrm{A} + (2\xi^2 - k^2)\,\mathrm{B}\}\,e^{i\xi x} \\ [\,p_{yy}]_0 &= \mu\,\{(2\xi^2 - k^2)\,\mathrm{A} + 2i\xi\beta\mathrm{B}\}\,e^{i\xi x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

The forms corresponding to (32) would be obtained by affixing accents to A and B, and reversing the signs of α, β.

4. In order to illustrate, and at the same time test, our method, it is convenient to begin with the solution of a known problem, viz., where a periodic force acts transversally on a line of matter, in an unlimited elastic solid.†

Let us imagine, in the first instance, that an extraneous force of amount $\mathrm{Y}e^{i\xi x}$ per unit area acts parallel to y on a thin stratum coincident with the plane $y = 0$. The normal stress will then be discontinuous at this plane, viz.,

$$[\,p_{yy}]_{y=+0} - [\,p_{yy}]_{y=-0} = -\,\mathrm{Y}e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

whilst the tangential stress is continuous. These conditions give, by (34),

$$\left.\begin{aligned} (2\xi^2 - k^2)(\mathrm{A} - \mathrm{A}') + 2i\xi\beta(\mathrm{B} + \mathrm{B}') &= -\,\frac{\mathrm{Y}}{\mu} \\ -\,2i\xi\alpha\,(\mathrm{A} + \mathrm{A}') + (2\xi^2 - k^2)\,(\mathrm{B} - \mathrm{B}') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (36).$$

Again, the continuity of u and v requires

$$\left.\begin{aligned} i\xi\,(\mathrm{A} - \mathrm{A}') - \beta\,(\mathrm{B} + \mathrm{B}') &= 0 \\ \alpha\,(\mathrm{A} + \mathrm{A}') + i\xi\,(\mathrm{B} - \mathrm{B}') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

---

* This convention should be carefully attended to; it runs throughout the paper.

† RAYLEIGH, 'Theory of Sound,' 2nd ed., § 376.

Hence

$$\mathrm{A} = -\,\mathrm{A}' = \frac{\mathrm{Y}}{2k^2\mu}, \quad \mathrm{B} = \mathrm{B}' = \frac{i\xi}{\beta}\cdot\frac{\mathrm{Y}}{2k^2\mu} \quad \ldots\ldots \quad (38).$$

We have, then, for $y > 0$,

$$\phi = \frac{\mathrm{Y}}{2k^2\mu}\,e^{-\alpha y}e^{i\xi x}, \quad \psi = \frac{\mathrm{Y}}{2k^2\mu}\cdot\frac{i\xi}{\beta}\,e^{-\beta y}e^{i\xi x} \quad \ldots\ldots \quad (39).$$

To pass to the case of an extraneous force Q concentrated on the line $x = 0$, $y = 0$, we make use of (19). Assuming that the $f(\lambda)$ of this formula vanishes for all but infinitesimal values of λ, for which it becomes infinite in such a way that

$$\int_{-\infty}^{\infty} f(\lambda)\,d\lambda = \mathrm{Q},$$

we write, in (39), $\mathrm{Y} = \mathrm{Q}\,d\xi/2\pi$, and integrate with respect to ξ from $-\infty$ to $+\infty$.* We thus obtain, for $y > 0$,

$$\phi = \frac{\mathrm{Q}}{4\pi k^2\mu}\int_{-\infty}^{\infty} e^{-\alpha y}e^{i\xi x}\,d\xi, \quad \psi = \frac{i\mathrm{Q}}{4\pi k^2\mu}\int_{-\infty}^{\infty}\frac{\xi e^{-\beta y}e^{i\xi x}\,d\xi}{\beta} \quad \ldots \quad (40),$$

or, on reference to (13),

$$\phi = -\frac{\mathrm{Q}}{4k^2\mu}\frac{\partial}{\partial y}\,\mathrm{D}_0(hr), \quad \psi = \frac{\mathrm{Q}}{4k^2\mu}\frac{\partial}{\partial x}\,\mathrm{D}_0(kr) \quad \ldots\ldots \quad (41),$$

where $r = \sqrt{(x^2 + y^2)}$.

If we put $x = r\cos\theta$, $y = r\sin\theta$, we find from (25), on inserting the time-factor, that for large values of r the radial and transverse displacements are

$$\left.\begin{aligned} \frac{\partial\phi}{\partial r} + \frac{\partial\psi}{r\,\partial\theta} &\doteq \frac{\mathrm{Q}}{4(\lambda + 2\mu)}\sqrt{\frac{2}{\pi hr}}\cdot e^{i(pt-hr-\frac{1}{4}\pi)}\sin\theta \\ \frac{\partial\phi}{r\,\partial\theta} - \frac{\partial\psi}{\partial r} &\doteq \frac{\mathrm{Q}}{4\mu}\sqrt{\frac{2}{\pi kr}}\cdot e^{i(pt-kr-\frac{1}{4}\pi)}\cos\theta \end{aligned}\right\} \quad \ldots\ldots \quad (42),$$

respectively.† Use has here been made of (7).

A simple expression can be obtained for the rate (W, say) at which the extraneous

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* The indeterminateness of the formula (19) in this case may be evaded by supposing, in the first instance, that the force Q, instead of being concentrated on the line $x = 0$, is uniformly distributed over the portion of the plane $y = 0$ lying between $x = \pm a$. It appears from (21) that we should then have

$$\mathrm{Y} = \frac{\mathrm{Q}}{2\pi}\cdot\frac{\sin\xi a}{\xi a}\,d\xi.$$

If we finally make $a = 0$ we obtain the results (40).

† The second of these results is equivalent to that given by RAYLEIGH, *loc. cit.*, for the case of incompressibility ($\lambda = \infty$).

force does work in generating the cylindrical waves which travel outwards from the source of disturbance. The formulæ (40) give, for the value of $\partial v/\partial t$ at the origin,

$$\frac{\partial v_0}{\partial t} = \frac{ipQe^{ipt}}{4\pi k^2\mu}\int_{-\infty}^{\infty}\left(\frac{\xi^2}{\beta} - \alpha\right)d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

This expression is really infinite, but we are only concerned with the part of it in the same phase with the force,* which is finite. Taking this alone, we have

$$\frac{\partial v_0}{\partial t} = \frac{pQe^{ipt}}{4\pi k^2\mu}\left\{\int_{-k}^{k}\frac{\xi^2 d\xi}{\sqrt{(k^2 - \xi^2)}} + \int_{-h}^{h}\sqrt{(h^2 - \xi^2)}\,d\xi\right\} = (k^2 + h^2)\,\frac{pQe^{ipt}}{8k^2\mu} \quad . \quad (44).$$

Discarding imaginary parts, we find that the mean rate, per unit length of the axis of z, at which a force Q cos pt does work is

$$W = \left(1 + \frac{h^2}{k^2}\right)\frac{pQ^2}{16\mu} = \frac{\lambda + 3\mu}{16\mu\,(\lambda + 2\mu)}\,pQ^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

5. We may proceed to the case of a "semi-infinite" elastic solid, bounded (say) by the plane $y = 0$, and lying on the positive side of this plane. We examine, in the first place, the effect of given periodic forces applied to the boundary.

As a typical distribution of *normal* force, we take

$$[p_{xy}]_0 = 0, \quad [p_{yy}]_0 = \mathrm{Y}e^{i\xi x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

the factor e^{ipt} being as usual understood. The constants A, B in (30) are determined by means of (34), viz.:

$$\left.\begin{aligned} -2i\xi\alpha\mathrm{A} + (2\xi^2 - k^2)\,\mathrm{B} &= 0,\\ (2\xi^2 - k^2)\,\mathrm{A} + 2i\xi\beta\mathrm{B} &= \frac{\mathrm{Y}}{\mu} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47).$$

Hence

$$\mathrm{A} = \frac{2\xi^2 - k^2}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Y}}{\mu}, \quad \mathrm{B} = \frac{2i\xi\alpha}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Y}}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48),$$

where, for shortness,

$$\mathrm{F}(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2\alpha\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

We shall find it convenient, presently, to write also

$$f(\xi) = (2\xi^2 - k^2)^2 + 4\xi^2\alpha\beta . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50).$$

---

* The awkwardness is evaded if (as in a previous instance) we distribute the force uniformly over a length $2a$ of the axis of x. This will introduce a factor $\left(\frac{\sin \xi a}{\xi a}\right)^2$ under the integral signs in the second member of (44).

The surface-values of the displacements are now given by (33), viz.:

$$\left.\begin{aligned} u_0 &= \frac{i\xi\,(2\xi^2 - k^2 - 2\alpha\beta)\,e^{i\xi x}}{\mathrm{F}(\xi)} \cdot \frac{\mathrm{Y}}{\mu}, \\ v_0 &= \frac{k^2\alpha e^{i\xi x}}{\mathrm{F}(\xi)} \cdot \frac{\mathrm{Y}}{\mu} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (51).$$

The effect of a concentrated force Q acting parallel to y at points of the line $x = 0$, $y = 0$ is deduced, as before, by writing $\mathrm{Y} = -\,\mathrm{Q}d\xi/2\pi$, and integrating from $-\infty$ to ∞; thus

$$\left.\begin{aligned} u_0 &= -\frac{i\mathrm{Q}}{2\pi\mu}\int_{-\infty}^{\infty} \frac{\xi\,(2\xi^2 - k^2 - 2\alpha\beta)\,e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)}, \\ v_0 &= -\frac{\mathrm{Q}}{2\pi\mu}\int_{-\infty}^{\infty} \frac{k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (52).$$

In a similar manner, corresponding to the *tangential* surface forces:

$$[p_{xy}]_0 = \mathrm{X}e^{i\xi x}, \quad [p_{yy}]_0 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (53),$$

we should find

$$\mathrm{A} = -\frac{2i\xi\beta}{\mathrm{F}(\xi)} \cdot \frac{\mathrm{X}}{\mu}, \quad \mathrm{B} = \frac{2\xi^2 - k^2}{\mathrm{F}(\xi)} \cdot \frac{\mathrm{X}}{\mu} \quad . \; . \; . \; . \; . \; . \; . \quad (54).$$

And, for the effect of a concentrated force P acting parallel to x at the origin,

$$\left.\begin{aligned} u_0 &= -\frac{\mathrm{P}}{2\pi\mu}\int_{-\infty}^{\infty} \frac{k^2\beta e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)}, \\ v_0 &= \frac{i\mathrm{P}}{2\pi\mu}\int_{-\infty}^{\infty} \frac{\xi\,(2\xi^2 - k^2 - 2\alpha\beta)\,e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (55).$$

The comparison of u_0 in (52) with v_0 in (55) gives an example of the general principle of reciprocity.*

We may also consider the case of an *internal* source of disturbance, resident (say) in the line $x = 0$, $y = f$, the boundary $y = 0$ being now entirely free. The simplest type of source is one which would produce symmetrical radial motion (in two dimensions) in an unlimited solid, say

$$\phi = \mathrm{D}_0\,(hr), \quad \psi = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (56),$$

where $r, = \sqrt{\{x^2 + (y - f)^2\}}$, denotes distance from the source. If we superpose on this an equal source in the line $x = 0$, $y = -f$, we obtain

$$\phi = \mathrm{D}_0\,(hr) + \mathrm{D}_0\,(hr'), \quad \psi = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (57),$$

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* RAYLEIGH, 'Theory of Sound,' vol. 1, § 108.

where $r' = \sqrt{\{x^2 + (y+f)^2\}}$. It is evident, without calculation, that the condition of zero tangential stress at the plane $y = 0$ is already satisfied; the normal stress, however, does not vanish. It appears from (13) that in the neighbourhood of the plane $y = 0$ the preceding value of ϕ is equivalent to

$$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha(y-f)}\, e^{i\xi x}\, d\xi}{\alpha} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha(y+f)}\, e^{i\xi x}\, d\xi}{\alpha}$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\cosh \alpha y}{\alpha}\, e^{-\alpha f}\, e^{i\xi x}\, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

Substituting in (29) we find that this makes

$$[p_{xy}]_0 = 0, \qquad [p_{yy}]_0 = \frac{2\mu}{\pi}\int_{-\infty}^{\infty} \frac{2\xi^2 - k^2}{\alpha}\, e^{-\alpha f}\, e^{i\xi x}\, d\xi \quad . \quad . \quad . \quad . \quad (59).$$

Comparing with (46), we see that the desired condition of zero stress on the boundary will be fulfilled, provided we superpose on (57) the solution obtained from (30) and (48) by putting

$$\mathrm{Y} = -\frac{2\mu}{\pi} \cdot \frac{2\xi^2 - k^2}{\alpha}\, e^{-\alpha f}\, d\xi,$$

and afterwards integrating with respect to ξ from $-\infty$ to ∞. The surface-displacements corresponding to this auxiliary solution are obtained from (51), and if we incorporate the part of u_0 due to (58), we find, after a slight reduction,

$$\left.\begin{aligned} u_0 &= -\frac{4ik^2}{\pi}\int_{-\infty}^{\infty} \frac{\beta\xi e^{-\alpha f}\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} \\ v_0 &= -\frac{2k^2}{\pi}\int_{-\infty}^{\infty} \frac{(2\xi^2 - k^2)\, e^{-\alpha f}\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

These calculations might be greatly extended. For example, it would be easy, with the help of Art. 4, to work out the case where a vertical or a horizontal periodic force acts on an internal line parallel to z. And, by means of the reciprocal theorem already adverted to, we could infer the horizontal or vertical displacement at an internal point due to a given localized surface force.

6. It remains to interpret, as far as possible, the definite integrals which occur in the expressions we have obtained.

It is to be remarked, in the first place, that the integrals, as they stand, are to a certain extent indeterminate, owing to the vanishing of the function $\mathrm{F}(\xi)$ for certain real values of ξ. It is otherwise evident *à priori* that on a particular solution of any of our problems we can superpose a system of free surface waves having the wave-length proper to the imposed period $2\pi/p$. The theory of such waves has been given by Lord RAYLEIGH,[*] and is moreover necessarily contained implicitly in our analysis.

Thus, if we put $Y = 0$ in (47), we find that the conditions of zero surface-stress are satisfied, provided

$$A : B = 2\kappa^2 - k^2 : 2i\kappa\alpha_1 = -2i\kappa\beta_1 : 2\kappa^2 - k^2 \quad . \quad . \quad . \quad . \quad (61),$$

where κ is a root of $F(\xi) = 0$, and α_1, β_1, denote the corresponding values of α, β, Now, in the notation of (49) and (50),

$$F(\xi) f(\xi) = (2\xi^2 - k^2)^4 - 16(\xi^2 - h^2)(\xi^2 - k^2)\xi^4$$
$$= k^8 \left\{1 - 8\frac{\xi^2}{k^2} + \left(24 - 16\frac{h^2}{k^2}\right)\frac{\xi^4}{k^4} - 16\left(1 - \frac{h^2}{k^2}\right)\frac{\xi^6}{k^6}\right\}. \quad . \quad (62).$$

Equating this to zero, we have a cubic in ξ^2/k^2, and since $k^2 > h^2$, it is plain that there is a real root between 1 and ∞. It may also be shown without much difficulty that the remaining roots, when real, lie between 0 and h^2/k^2. The former root makes α, β real and positive, and therefore cannot make $f(\xi) = 0$. The latter roots make α, β positive imaginaries, and therefore cannot make $F(\xi) = 0$. This latter equation has accordingly only two real roots $\xi = \pm\kappa$, where $\kappa > k$.

Thus, in the case of incompressibility ($\lambda = \infty$, $h = 0$) it is found that

$$\kappa/k = 1{\cdot}04678\ldots,$$

and that the remaining roots of (62) are complex.[†] On POISSON's hypothesis as to the relation between the elastic constants ($\lambda = \mu$, $h^2 = \frac{1}{3}k^2$), the roots of (62) are all real, viz., they are

$$\xi^2/k^2 = \tfrac{1}{4}, \quad \tfrac{1}{4}(3 - \sqrt{3}), \quad \tfrac{1}{4}(3 + \sqrt{3}),$$

so that

$$\kappa/k = \tfrac{1}{2}\sqrt{(3 + \sqrt{3})} = 1{\cdot}087664\ldots;$$

this will usually be taken as the standard case for purposes of numerical illustration.

In analogy with (28), it will be convenient to write

$$\kappa = pc \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

where c denotes the wave-slowness of the Rayleigh waves. The corresponding wave-velocity is

$$c^{-1} = \frac{k}{\kappa} \cdot b^{-1} = \frac{k}{\kappa} \cdot \sqrt{\frac{\mu}{\rho}}.$$

According as we suppose $\lambda = \infty$, or $\lambda = \mu$, this is ·9553 times, or ·9194 times, the velocity of propagation of plane transverse waves in an unlimited solid.

The further properties of free Rayleigh waves are contained in the formulæ (61) and (30). We merely note, for purposes of reference, that if in (33) we put $\xi = \pm \kappa$, and accordingly, from (61),

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[*] 'Proc. Lond. Math. Soc.,' vol. 17 (1885); 'Scientific Papers,' vol. 2, p. 441.

[†] *Cf.* RAYLEIGH (*loc. cit.*), where it is also shown (virtually) that they are roots of $f(\xi)$, not of $F(\xi)$, if α, β be chosen so as to have their real parts positive.

$$A = (2\kappa^2 - k^2)\,C, \quad B = \pm\, 2i\kappa\alpha_1 C \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

we obtain by superposition a system of standing waves in which

$$u_0 = -\,2\kappa\,(2\kappa^2 - k^2 - 2\alpha_1\beta_1)\,C \sin \kappa x \,.\, e^{ipt}, \quad v_0 = 2k^2\alpha_1 C \cos \kappa x \,.\, e^{ipt} \quad . \quad (65).$$

The theory here recapitulated indicates the method to be pursued in treating the definite integrals of Art. 5. We fix our attention, in the first instance, on their "principal values," in CAUCHY's sense, and afterwards superpose such a system of free Rayleigh waves as will make the final result consist solely of waves travelling outwards from the origin of disturbance.

It may be remarked that an alternative procedure is possible, in which even temporary indeterminateness is avoided. This consists in inserting in the equations of motion (23) frictional terms proportional to the velocities, and finally making the coefficients of these terms vanish. This method has some advantages, especially as regards the positions of the "singular points" to be referred to. The chief problem of this paper was, in fact, first worked through in this manner; but as the method seemed rather troublesome to expound as regards some points of detail, it was abandoned in favour of that explained above.

7. The most important case, and the one here chiefly considered, is that of a concentrated *vertical* force applied to the surface, to which the formulæ (52) relate. The case of a *horizontal* force, expressed by the formulæ (55), could be treated in an exactly similar manner.

Since u_0 is evidently an odd, and v_0 an even, function of x, it will be sufficient to take the case of x positive.

As regards the horizontal* displacement u_0, we consider the integral

$$\int \Phi(\zeta)\,d\zeta = \int \frac{\zeta\,\{(2\zeta^2 - k^2) - 2\sqrt{(\zeta^2 - h^2)}\,\sqrt{(\zeta^2 - k^2)}\}\,e^{i\zeta x}\,d\zeta}{(2\zeta^2 - k^2)^2 - 4\sqrt{(\zeta^2 - h^2)}\,\sqrt{(\zeta^2 - k^2)}\,\zeta^2} \quad . \quad . \quad (66),$$

taken round a suitable contour in the plane of the complex variable $\zeta, = \xi + i\eta$. If this contour does not include either "poles" $(\pm\,\kappa,\ 0)$, or "branch-points" $(\pm\,h,\ 0)$, $(\pm\,k,\ 0)$ of the function to be integrated, the result will be zero.

A convenient contour for our purpose is a rectangle, one side of which consists of the axis of ξ except for small semicircular indentations surrounding the singular points specified, whilst the remaining sides are at an infinite distance on the side $\eta > 0$. It is easily seen that the parts of the integral due to these infinitely distant sides will vanish of themselves. If we adopt for the radicals $\sqrt{(\zeta^2 - h^2)}$ and $\sqrt{(\zeta^2 - k^2)}$,

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* The sense in which the terms "horizontal" and "vertical" are used is indicated in the second sentence of the Introduction.

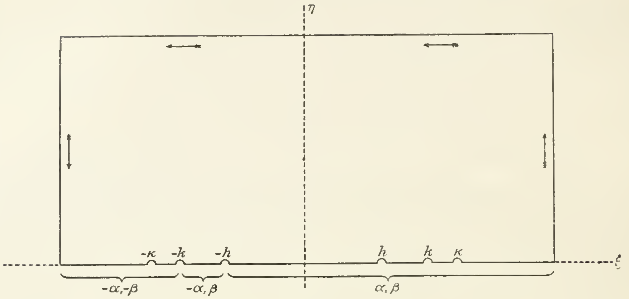

Fig. 1.

at points of the axis of ξ, the consistent system of values indicated in fig. 1,* we find, for the various parts of the first-mentioned side,†

$$\int_{-\infty}^{-k} \Phi(\zeta)\, d\zeta = \mathfrak{P}\int_{-\infty}^{-k} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} - i\pi \frac{-\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1)\, e^{-i\kappa x}}{\mathrm{F}'(-\kappa)},$$

$$\int_{-k}^{-h} \Phi(\zeta)\, d\zeta = \int_{-k}^{-h} \frac{\xi(2\xi^2 - k^2 + 2\alpha\beta)\, e^{i\xi x}\, d\xi}{f(\xi)},$$

$$\int_{-h}^{\infty} \Phi(\zeta)\, d\zeta = \mathfrak{P}\int_{-h}^{\infty} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} - i\pi \frac{\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1)\, e^{i\kappa x}}{\mathrm{F}'(\kappa)},$$

where the terms with $\mathrm{F}'(-\kappa)$ and $\mathrm{F}'(\kappa)$ in the denominator are due to the small semicircles about the points $(\pm\kappa, 0)$. Equating the sum of these expressions to zero, we find, since $\mathrm{F}'(-\kappa) = -\mathrm{F}'(\kappa)$,

$$\begin{aligned} \mathfrak{P}\int_{-\infty}^{\infty} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\, e^{i\xi x}\, d\xi}{\mathrm{F}(\xi)} &= -2i\pi \mathrm{H} \cos\kappa x \\ &\quad + \int_{-k}^{-h} \left\{ \frac{2\xi^2 - k^2 - 2\alpha\beta}{\mathrm{F}(\xi)} - \frac{2\xi^2 - k^2 + 2\alpha\beta}{f(\xi)} \right\} \xi e^{i\xi x}\, d\xi \\ &= -2i\pi \mathrm{H}\cos\kappa x - 4k^2 \int_h^k \frac{\xi(2\xi^2 - k^2)\, \alpha\beta e^{-i\xi x}\, d\xi}{\mathrm{F}(\xi) f(\xi)} \quad . \quad . \quad . \quad . \quad (67), \end{aligned}$$

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* The function under the integral sign in (66) is uniquely determined (by continuity) within and on the contour when once the values of the radicals $\surd(\zeta^2 - h^2)$ and $\surd(\zeta^2 - k^2)$ at some one point are assigned. The convention implied in the text is that the radicals are both positive at the point $(+\infty, 0)$.

It will be noticed that over the portion of the axis of ξ between $-k$ and $-h$ the function in (66) differs from that involved in the value of u_0 as given by (52). This is allowed for in the second member of (67). Corrections, or rather adjustments, of this kind occur repeatedly in the transformations of this paper.

† The symbol $\mathfrak{P}$ is used to distinguish the "principal value" of an integral (with respect to a real variable) to which it is prefixed.

where

$$\mathrm{H} = -\frac{\kappa\,(2\kappa^2 - k^2 - 2\alpha_1\beta_1)}{\mathrm{F}'(\kappa)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68),$$

a numerical quantity depending only on the ratio $\lambda : \mu$.

To examine the value of v_0 we take the integral

$$\int \Psi(\zeta)\,d\zeta = \int \frac{k^2\sqrt{(\zeta^2 - h^2)}\,e^{i\zeta x}\,d\zeta}{(2\zeta^2 - k^2)^2 - 4\sqrt{(\zeta^2 - h^2)}\sqrt{(\zeta^2 - k^2)}\,\zeta^2} \quad . \quad . \quad . \quad (69)$$

round the same contour. Integrating along the axis of ξ we find

$$\int_{-\infty}^{-k} \Psi(\zeta)\,d\zeta = \mathfrak{P}\int_{-\infty}^{-k} \frac{-k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} - i\pi\,\frac{-k^2\alpha_1}{\mathrm{F}'(-\kappa)}\,e^{-i\kappa x},$$

$$\int_{-k}^{-h} \Psi(\zeta)\,d\zeta = \int_{-k}^{-h} \frac{-k^2\alpha e^{i\xi x}\,d\xi}{f(\xi)},$$

$$\int_{-h}^{\infty} \Psi(\zeta)\,d\zeta = \mathfrak{P}\int_{-h}^{\infty} \frac{k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} - i\pi\,\frac{k^2\alpha_1}{\mathrm{F}'(\kappa)}\,e^{i\kappa x},$$

and thence by addition, since the terms due to the infinitely distant parts of the contour vanish as before,

$$\begin{aligned} \mathfrak{P}\int_{-\infty}^{\infty} \frac{k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} &= -2i\pi\mathrm{K}\cos\kappa x + \mathfrak{P}\int_{-\infty}^{-k} \frac{2k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} \\ &\quad + \int_{-k}^{-h} \left\{\frac{1}{\mathrm{F}(\xi)} + \frac{1}{f(\xi)}\right\} k^2\alpha e^{i\xi x}\,d\xi \\ &= -2i\pi\mathrm{K}\cos\kappa x + 2\mathfrak{P}\int_{k}^{\infty} \frac{k^2\alpha e^{-i\xi x}\,d\xi}{\mathrm{F}(\xi)} \\ &\quad + 2k^2\int_{h}^{k} \frac{(2\xi^2 - k^2)^2\,\alpha e^{-i\xi x}\,d\xi}{\mathrm{F}(\xi)\,f(\xi)} \quad . \quad . \quad . \quad . \quad . \quad (70), \end{aligned}$$

where

$$\mathrm{K} = -\frac{k^2\alpha_1}{\mathrm{F}'(\kappa)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

Hence if to the principal values of the expressions in (52) we add the system of free Rayleigh waves,

$$u_0 = i\,\frac{\mathrm{Q}}{\mu}\,\mathrm{H}\sin\kappa x, \qquad v_0 = -i\,\frac{\mathrm{Q}}{\mu}\,\mathrm{K}\cos\kappa x \quad . \quad . \quad . \quad . \quad . \quad (72),$$

which is evidently of the type (65), we obtain, on inserting the time-factor,

$$u_0 = -\frac{Q}{\mu} \mathrm{H} e^{i(pt-\kappa x)} - \frac{2Q}{\pi\mu}\int_h^k \frac{k^2\xi(2\xi^2-k^2)\sqrt{(\xi^2-h^2)}\sqrt{(k^2-\xi^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^4+16\xi^4(\xi^2-h^2)(k^2-\xi^2)} \quad (73),$$

$$v_0 = -\frac{Q}{\pi\mu}\mathfrak{P}\int_k^\infty \frac{k^2\sqrt{(\xi^2-h^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^2-4\xi^2\sqrt{(\xi^2-h^2)}\sqrt{(\xi^2-k^2)}}$$

$$-\frac{Q}{\pi\mu}\int_h^k \frac{k^2(2\xi^2-k^2)^2\sqrt{(\xi^2-h^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^4+16\xi^4(\xi^2-h^2)(k^2-\xi^2)} \quad . \quad . \quad . \quad (74).$$

This is for x positive; the corresponding results for x negative would be obtained by changing the sign of x in the exponentials, and reversing the sign of u_0.

The solution thus found is made up of waves travelling outwards, right and left, from the origin, and so satisfies all the conditions of the question.

The first term in u_0 gives, on each side, a train of waves travelling unchanged with the velocity c^{-1}. The second term gives an aggregate of waves travelling with velocities ranging from b^{-1} to a^{-1}. As x is increased, this term diminishes indefinitely, owing to the more and more rapid fluctuations in the value of $e^{i\xi x}$.

On the other hand, the part of v_0 which corresponds to the first term of u_0 remains embedded in the first definite integral in (74). To disentangle it we must have recourse to another treatment of the integral $\int\Psi(\zeta)\,d\zeta$. One way of doing this is to take the integral round the pair of contours shown in fig. 2, where a consistent scheme of

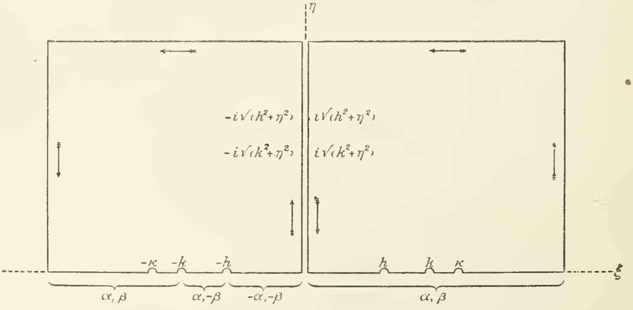

Fig. 2.

values to be attributed to the radicals $\sqrt{(\zeta^2-h^2)}$ and $\sqrt{(\zeta^2-k^2)}$ is indicated. For the only parts of the left-hand contour which need be taken into account we find

$$\int_{-\infty}^{-k} \Psi(\zeta)\,d\zeta = \mathfrak{P}\int_{-\infty}^{-k} \frac{k^2 \alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} - i\pi \frac{k^2 \alpha_1}{\mathrm{F}'(-\kappa)} e^{-i\kappa x},$$

$$\int_{-k}^{-h} \Psi(\zeta)\,d\zeta = \int_{-k}^{-h} \frac{k^2 \alpha e^{i\xi x}\,d\xi}{f(\xi)},$$

$$\int_{-h}^{0} \Psi(\zeta)\,d\zeta = \int_{-h}^{0} \frac{-k^2 \alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)},$$

$$\int_{0}^{i\infty} \Psi(\zeta)\,d\zeta = \int_{0}^{\infty} \frac{-i\sqrt{(h^2+\eta^2)}\,k^2 e^{-\eta x} i\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}}.$$

Similarly, in the right-hand contour,

$$\int_{i\infty}^{0} \Psi(\zeta)\,d\zeta = \int_{\infty}^{0} \frac{i\sqrt{(h^2+\eta^2)}\,k^2 e^{-\eta x} i\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}},$$

$$\int_{0}^{\infty} \Psi(\zeta)\,d\zeta = \mathfrak{P}\int_{0}^{\infty} \frac{k^2 \alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} - i\pi \frac{k^2 \alpha_1}{\mathrm{F}'(\kappa)} e^{i\kappa x}.$$

We infer, by addition,

$$\mathfrak{P}\int_{-\infty}^{\infty} \frac{k^2 \alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} = 2\pi \mathrm{K} \sin \kappa x + 2\int_{0}^{h} \frac{k^2 \alpha e^{-i\xi x}\,d\xi}{\mathrm{F}(\xi)}$$
$$+ \int_{h}^{k} \left\{\frac{1}{\mathrm{F}(\xi)} - \frac{1}{f(\xi)}\right\} k^2 \alpha e^{-i\xi x}\,d\xi - 2\int_{0}^{\infty} \frac{k^2\sqrt{(h^2+\eta^2)}\,e^{-\eta x}\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}} \quad (75).$$

If we multiply this by $-\mathrm{Q}/2\pi\mu$, and add in the term due to the free Rayleigh waves represented by (72), we obtain, as an equivalent form of (74),

$$v_0 = -\frac{i\mathrm{Q}}{\mu} \mathrm{K} e^{i(pt-\kappa x)} - \frac{i\mathrm{Q}}{\pi\mu}\int_{0}^{h} \frac{k^2\sqrt{(h^2-\xi^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^2 + 4\xi^2\sqrt{(h^2-\xi^2)}\sqrt{(k^2-\xi^2)}}$$
$$- \frac{4i\mathrm{Q}}{\pi\mu}\int_{h}^{k} \frac{k^2\xi^2(\xi^2-h^2)\sqrt{(k^2-\xi^2)}\,e^{i(pt-\xi x)}\,d\xi}{(2\xi^2-k^2)^4 + 16\xi^4(\xi^2-h^2)(k^2-\xi^2)}$$
$$+ \frac{\mathrm{Q}}{\pi\mu} e^{ipt}\int_{0}^{\infty} \frac{k^2\sqrt{(h^2+\eta^2)}\,e^{-\eta x}\,d\eta}{(2\eta^2+k^2)^2 - 4\eta^2\sqrt{(h^2+\eta^2)}\sqrt{(k^2+\eta^2)}} \quad (76).^*$$

It is evident that all terms after the first diminish indefinitely as x is increased.

---

* From this we can deduce, by the same method as in Art. 4, an expression for the mean rate W at which a vertical pressure Q cos pt does work in generating waves, viz.,

$$\mathrm{W} = \tfrac{1}{2}\mathrm{K}\frac{p\mathrm{Q}^2}{\mu} + \frac{p\mathrm{Q}^2}{2\pi\mu}\int_{0}^{h} \frac{k^2\sqrt{(h^2-\xi^2)}\,d\xi}{(2\xi^2-k^2)^2 + 4\xi^2\sqrt{(h^2-\xi^2)}\sqrt{(k^2-\xi^2)}}$$
$$+ \frac{2p\mathrm{Q}^2}{\pi\mu}\int_{h}^{k} \frac{k^2\xi^2(\xi^2-h^2)\sqrt{(k^2-\xi^2)}\,d\xi}{(2\xi^2-k^2)^4 + 16\xi^4(\xi^2-h^2)(k^2-\xi^2)}.$$

If in (73) and (76) we regard only the terms which are sensible at a great distance from the origin, we have, for x positive,

$$u_0 = -\frac{Q}{\mu} \mathrm{H} e^{ip(t-cx)}, \qquad v_0 = -i\frac{Q}{\mu} \mathrm{K} e^{ip(t-cx)} \quad . \quad . \quad . \quad . \quad . \quad (77);$$

and similarly for x negative we should find

$$u_0 = \frac{Q}{\mu} \mathrm{H} e^{ip(t+cx)}, \quad v_0 = -i\frac{Q}{\mu} \mathrm{K} e^{ip(t+cx)} \quad . \quad . \quad . \quad . \quad . \quad (78).$$

These formulæ represent a system of free Rayleigh waves, except for the discontinuity at the origin, where the extraneous force is applied. The vibrations are elliptic, with horizontal and vertical axes in the ratio of the two numbers H and K, which are defined by (68) and (71), respectively. To calculate these, we have, since $\mathrm{F}(\kappa) = 0$,

$$f(\kappa) = 2(2\kappa^2 - k^2)^2 = 8\alpha_1\beta_1\kappa^2,$$

and therefore

$$\mathrm{H} = \frac{k^2(2\kappa^2 - k^2)^3}{-\kappa \mathrm{F}'(\kappa) f(\kappa)}, \quad \mathrm{K} = \frac{2k^2\alpha_1(2\kappa^2 - k^2)^2}{-\mathrm{F}'(\kappa) f(\kappa)} \quad . \quad . \quad . \quad . \quad . \quad (79),$$

where, by differentiation of (62),

$$-\mathrm{F}'(\kappa) f(\kappa) = 16k^6\kappa \left\{1 - \left(6 - 4\frac{h^2}{k^2}\right)\frac{\kappa^2}{k^2} + 6\left(1 - \frac{h^2}{k^2}\right)\frac{\kappa^4}{k^4}\right\} \quad . \quad . \quad (80).$$

In the case of incompressibility I find

$$\mathrm{H} = \cdot 05921, \quad \mathrm{K} = \cdot 10890;$$

whilst on POISSON'S hypothesis

$$\mathrm{H} = \cdot 12500, \quad \mathrm{K} = \cdot 18349,$$

so that the amplitudes are, for the same value of μ and for the same applied force, about double what they are in the case of incompressibility.

A similar treatment applies to the formulæ (55), which represent the effect of a concentrated *horizontal* force $\mathrm{P}e^{ipt}$. Taking account only of the more important terms, I find, for x positive,

$$u_0 = -\frac{i\mathrm{P}}{\mu} \mathrm{H}' e^{ip(t-cx)}, \quad v_0 = \frac{\mathrm{P}}{\mu} \mathrm{K}' e^{ip(t-cx)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (81),$$

and, for x negative,

$$u_0 = -\frac{i\mathrm{P}}{\mu} \mathrm{H}' e^{ip(t+cx)}, \quad v_0 = -\frac{\mathrm{P}}{\mu} \mathrm{K}' e^{ip(t+cx)} \quad . \quad . \quad . \quad . \quad . \quad (82),$$

where

$$\left.\begin{aligned} \mathrm{H}' &= -\frac{k^2\beta_1}{\mathrm{F}'(\kappa)} = \frac{2k^2\beta_1(2\kappa^2 - k^2)^2}{-\mathrm{F}'(\kappa) f(\kappa)} \\ \mathrm{K}' &= -\frac{\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1)}{\mathrm{F}'(\kappa)} = \frac{k^2(2\kappa^2 - k^2)^3}{-\kappa \mathrm{F}'(\kappa) f(\kappa)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (83).$$

The ratio of H' to K' is, of course, equal to that of H to K; K' is, moreover, identical with H, in conformity with the principle of reciprocity already referred to. It appears, therefore, from the numerical values of H, K above given, that for $\lambda = \infty$

$$H' = \cdot 03219, \quad K' = \cdot 05921;$$

and for $\lambda = \mu$

$$H' = \cdot 08516, \quad K' = \cdot 12500.$$

Again, in the case of the internal source (56) I find, for large positive values of x,

$$u_0 = -8\kappa H' e^{-a_1 f} e^{ip(t-cx)}, \quad v_0 = 8i\kappa K' e^{-a_1 f} e^{ip(t-cx)} \quad . \quad . \quad . \quad . \quad . \quad (84),$$

and, for large negative values,

$$u_0 = 8\kappa H' e^{-a_1 f} e^{ip(t+cx)}, \quad v_0 = 8i\kappa K' e^{-a_1 f} e^{ip(t+cx)} \quad . \quad . \quad . \quad . \quad . \quad (85).$$

The factor $e^{-a_1 f}$ indicates how the surface effect (at a sufficient distance) varies with the depth of the source.

8. If in any of the preceding cases we wish to examine more closely the nature and magnitude of the residual disturbance, so far as it is manifested at the surface, it is more convenient to use the system of contours shown in fig. 3. With this system we

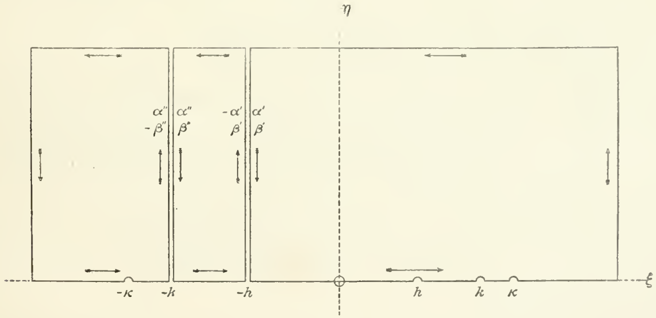

Fig. 3.

can so adjust matters that the radicals $\sqrt{(\zeta^2 - h^2)}$ and $\sqrt{(\zeta^2 - k^2)}$ shall assume in all parts of the axis of ξ exactly the values α, β with which we are concerned in formulæ such as (52). It is convenient, for brevity, to denote by $\pm\alpha'$, β' the values assumed by the same radicals on the two sides of the lines $\xi = -h$, and by α'', $\pm\beta''$ their values on the two sides of the line $\xi = -k$, these values being supposed determined in accordance with the requirements of continuity. Thus, with the allocation shown in the figure, we shall have, for small values of η,

$$\left.\begin{array}{ll}\alpha' = -\sqrt{(2h\eta)}e^{-\frac{1}{4}i\pi}, & \beta' = i\sqrt{(k^2-h^2)} \\ \alpha'' = \sqrt{(k^2-h^2)}, & \beta'' = -\sqrt{(2k\eta)}\,e^{-\frac{1}{4}i\pi}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (86),$$

approximately.

Taking the integral (66) round the several contours, in the directions shown by the arrows, we find

$$\mathfrak{P}\int_{-\infty}^{\infty}\frac{\xi(2\xi^2-k^2-2\alpha\beta)\,e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} = -2i\pi\mathrm{H}\cos\kappa x$$

$$+\,e^{-ikx}\int_0^\infty\left\{\frac{2\zeta^2-k^2-2\alpha''\beta''}{(2\zeta^2-k^2)^2-4\zeta^2\alpha''\beta''}-\frac{2\zeta^2-k^2+2\alpha''\beta''}{(2\zeta^2-k^2)^2+4\zeta^2\alpha''\beta''}\right\}\zeta e^{-\eta x}i\,d\eta$$

$$+\,e^{-ihx}\int_0^\infty\left\{\frac{2\zeta^2-k^2-2\alpha'\beta'}{(2\zeta^2-k^2)^2-4\zeta^2\alpha'\beta'}-\frac{2\zeta^2-k^2+2\alpha'\beta'}{(2\zeta^2-k^2)^2+4\zeta^2\alpha'\beta'}\right\}\zeta e^{-\eta x}i\,d\eta\,,$$

$$= -2i\pi\mathrm{H}\cos\kappa x + 4ie^{-ikx}\int_0^\infty\frac{k^2(2\zeta^2-k^2)\,\alpha''\beta''\zeta e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)}$$

$$+\,4ie^{-ihx}\int_0^\infty\frac{k^2(2\zeta^2-k^2)\,\alpha'\beta'\zeta e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)} \quad . \quad . \quad (87),$$

where, in the first integral, $\zeta = -k + i\eta$, and, in the second, $\zeta = -h + i\eta$.

The integral (69), taken round the same contours, gives

$$\mathfrak{P}\int_{-\infty}^{\infty}\frac{k^2\alpha e^{i\xi x}\,d\xi}{\mathrm{F}(\xi)} = 2\pi\mathrm{K}\sin\kappa x + e^{-ikx}\int_0^\infty\left\{\frac{k^2\alpha''}{(2\zeta^2-k^2)^2-4\zeta^2\alpha''\beta''}-\frac{k^2\alpha''}{(2\zeta^2-k^2)^2+4\zeta^2\alpha''\beta''}\right\}e^{-\eta x}i\,d\eta$$

$$+\,e^{-ihx}\int_0^\infty\left\{\frac{k^2\alpha'}{(2\zeta^2-k^2)^2-4\zeta\alpha'\beta'}-\frac{-k^2\alpha'}{(2\zeta^2-k^2)^2+4\zeta^2\alpha'\beta'}\right\}e^{-\eta x}i\,d\eta$$

$$= 2\pi\mathrm{K}\sin\kappa x + 8ie^{-ikx}\int_0^\infty\frac{k^2\zeta^2(\zeta^2-h^2)\,\beta''e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)}$$

$$+\,2ie^{-ihx}\int_0^\infty\frac{k^2(2\zeta^2-k^2)^2\alpha' e^{-\eta x}\,d\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)} \quad . \quad . \quad . \quad (88),$$

on the same understanding.

The definite integrals in these results can all be expanded in asymptotic forms by means of the formula

$$\int_0^\infty\eta^{\frac{1}{2}}\chi(\eta)\,e^{-\eta x}\,d\eta = \frac{\Pi\left(\frac{1}{2}\right)}{x^{\frac{3}{2}}}\chi(0) + \frac{\Pi\left(\frac{3}{2}\right)}{x^{\frac{5}{2}}}\frac{\chi'(0)}{1\,!} + \frac{\Pi\left(\frac{5}{2}\right)}{x^{\frac{7}{2}}}\frac{\chi''(0)}{2\,!} + \ldots \quad . \quad (89);$$

and when hx, and therefore also kx, is sufficiently large, the first terms in the expansions will give an adequate approximation.

Thus, taking account of (86), the last members of (87) and (88) are equivalent to

$$-2i\pi \mathrm{H} \cos \kappa x + 2\sqrt{(2\pi)} \sqrt{\left(1 - \frac{h^2}{k^2}\right)} \cdot \frac{ie^{-i(kx+\frac{1}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$+ 2\sqrt{(2\pi)} \frac{h^3k^2 \sqrt{(k^2 - h^2)}}{(k^2 - 2h^2)^3} \cdot \frac{e^{-i(hx+\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c.,$$

and

$$2\pi \mathrm{K} \sin \kappa x - 4\sqrt{(2\pi)} \left(1 - \frac{h^2}{k^2}\right) \cdot \frac{ie^{-i(kx+\frac{1}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$- \sqrt{(2\pi)} \frac{h^2k^2}{(k^2 - 2h^2)^2} \cdot \frac{ie^{i-(hx+\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c.,$$

respectively. Substituting in (52), and adding in the system (72) as before, we have, for large positive values of x,

$$u_0 = -\frac{\mathrm{Q}}{\mu} \mathrm{H}e^{i(pt-\kappa x)} + \frac{\mathrm{Q}}{\mu} \sqrt{\frac{2}{\pi}} \sqrt{\left(1 - \frac{h^2}{k^2}\right)} \cdot \frac{e^{i(pt-kx-\frac{1}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$- \frac{\mathrm{Q}}{\mu} \sqrt{\frac{2}{\pi}} \cdot \frac{h^3k^2 \sqrt{(k^2 - h^2)}}{(k^2 - 2h^2)^3} \cdot \frac{ie^{i(pt-hx-\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c. \quad . \quad (90),$$

$$v_0 = -\frac{i\mathrm{Q}}{\mu} \mathrm{K}e^{i(pt-\kappa x)} + \frac{2\mathrm{Q}}{\mu} \sqrt{\frac{2}{\pi}} \cdot \left(1 - \frac{h^2}{k^2}\right) \cdot \frac{ie^{i(pt-kx-\frac{1}{4}\pi)}}{(kx)^{\frac{3}{2}}}$$

$$+ \frac{\mathrm{Q}}{2\mu} \sqrt{\frac{2}{\pi}} \cdot \frac{h^2k^2}{(k^2 - 2h^2)^2} \cdot \frac{ie^{i(pt-hx-\frac{1}{4}\pi)}}{(hx)^{\frac{3}{2}}} + \&c. \quad . \quad . \quad . \quad (91).$$

The first terms in these expressions have already been interpreted. The residual disturbance constitutes a sort of fringe to the cylindrical elastic waves which are propagated into the interior of the solid, and consists of two parts. In one of these the wave-velocity p/k, or b^{-1}, is that of equivoluminal waves; the vibrations (at the surface) are elliptic, the ratio of the vertical to the horizontal diameter of the orbit being $2\sqrt{(1 - h^2/k^2)}$, or 1·633 for $\lambda = \mu$. The remaining part has the wave-velocity p/h, or a^{-1}, of irrotational waves; the surface vibrations which it represents are rectilinear, the ratio of the vertical to the horizontal amplitude being $(k^2 - 2h^2)/2h\,(k^2 - h^2)^{\frac{1}{2}}$, or ·3535 for $\lambda = \mu$. With increasing distance x the amplitude of each part diminishes as $x^{-\frac{3}{2}}$, whereas in an unlimited solid the law is $x^{-\frac{1}{2}}$, as appears from (42).

Similar results will obviously hold in the case of the other problems considered in Art. 5.

9. It has been assumed, up to this stage, that the primary disturbance varies as a simple-harmonic function of the time. It is proposed now to generalize the law of variation, and in particular to examine the effect of a single impulse of short duration. From this the general case can be inferred by superposition.

It is to be noticed, in all our formulæ, that if we write

$$\xi = p\theta, \quad h = pa, \quad k = pb, \quad \kappa = pc,$$

the symbol p which determines the frequency will disappear, except in the exponentials; this greatly facilitates the desired generalization by means of FOURIER'S theorem. Thus, in the case of a concentrated vertical pressure $Q(t)$ acting on the surface, the formulæ (73) and (74) lead to

$$u_0 = -\frac{\mathrm{H}}{\mu}\mathrm{Q}(t-cx) - \frac{2}{\pi\mu}\int_a^b \frac{b^2\theta(2\theta^2-b^2)\sqrt{(\theta^2-a^2)}\sqrt{(b^2-\theta^2)}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)} \cdot \mathrm{Q}(t-\theta x)\,d\theta \ . \quad (92),$$

$$v_0 = -\frac{1}{\pi\mu}\int_a^b \frac{b^2(2\theta^2-b^2)^2\sqrt{(\theta^2-a^2)}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)} \cdot \mathrm{Q}(t-\theta x)\,d\theta$$

$$- \frac{1}{\pi\mu}\mathfrak{P}\int_b^\infty \frac{b^2\sqrt{(\theta^2-a^2)}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{(\theta^2-a^2)}\sqrt{(\theta^2-b^2)}} \cdot \mathrm{Q}(t-\theta x)\,d\theta \ . \quad (93).$$

The definite integrals represent aggregates of waves, of the same general type, travelling with slownesses ranging from a to b, and from b to ∞, respectively.

If we suppose that $\mathrm{Q}(t)$ vanishes for all but small values of t, it appears from (92) that the horizontal disturbance at a distance x begins (as we should expect) after a time ax, which is the time a wave of expansion would take to travel the distance; it lasts till a time bx, which is the time distortional waves would take to travel the distance; and then, for a while, ceases.* Finally, about the time cx, comes a solitary wave of short duration (the same as that of the primary impulse) represented by the first term of (92). This wave is of unchanging type, whereas the duration of the preliminary disturbance varies directly as x, and its amplitude (as will be seen immediately) varies inversely as x.

If we put

$$\bar{\mathrm{Q}} = \int \mathrm{Q}(t)\,dt \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (94),$$

the integration extending over the short range for which Q is sensible, the preliminary horizontal disturbance will be given by

$$u_0 = \frac{2\bar{\mathrm{Q}}}{\pi\mu bx} \cdot \mathrm{U}\left(\frac{t}{x}\right) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (95),$$

provided

$$\mathrm{U}(\theta) = -\frac{b^3\theta(2\theta^2-b^2)\sqrt{(\theta^2-a^2)}\sqrt{(b^2-\theta^2)}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)} \quad . \ . \ . \ . \quad (96),$$

where $a < \theta < b$. The following table gives the values of $\mathrm{U}(\theta)$ for a series of values of θ/a, on the hypothesis of $\lambda = \mu$, or $b/a = 1{\cdot}7321$.

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* This temporary cessation of the horizontal motion is special to the case of a normal impulse. If the impulse be tangential, the contrast between the horizontal and vertical motions, in this respect, is reversed.

θ/a.	U (θ).	θ/a.	U (θ).	θ/a.	U (θ).	θ/a.	U (θ).
1·000	0	1·025	+ ·62777	1·10	+ ·22789	1·550	− ·15122
1·001	+ ·31247	1·030	+ ·59351	1·15	+ ·10295	1·600	− ·15842
1·002	+ ·42080	1·035	+ ·55806	1·20	+ ·02722	1·625	− ·15927
1·003	+ ·49148	1·040	+ ·52308	1·25	− ·02311	1·650	− ·15681
1·004	+ ·54191	1·050	+ ·45741	1·30	− ·05905	1·675	− ·14845
1·005	+ ·57926	1·060	+ ·39889	1·35	− ·08622	1·700	− ·12795
1·010	+ ·66493	1·070	+ ·34746	1·40	− ·10771	1·725	− ·07021
1·015	+ ·67536	1·080	+ ·30238	1·45	− ·12527	b/a	0
1·020	+ ·65744	1·090	+ ·26279	1·50	− ·13975	—	—

The function has a maximum value $+ {\cdot}67643$ when $\theta/a = 1{\cdot}01368$; it changes sign when $\theta/a = 1{\cdot}22474$; and it has a minimum value $- {\cdot}159319$ when $\theta/a = 1{\cdot}62076$.* The graph of this function is shown in the upper part of fig. 4. If the scales be properly chosen, the curve will represent the variation of u_0 with t, during the "preliminary" disturbance, at any assigned point x. For this purpose the horizontal scale must vary directly, and the vertical scale inversely, as x.

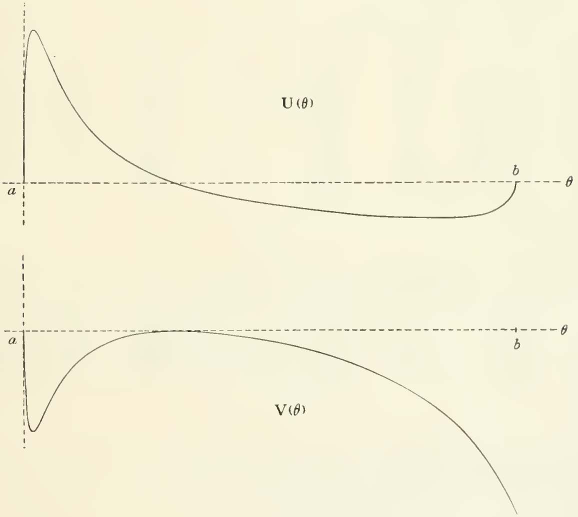

Fig. 4.

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* The calculations were made almost entirely by Mr. H. J. Woodall, to whom I am much indebted.

The interpretation of the expression (93) for the vertical displacement v_0 is not quite so simple. For a given value of x, the most important part is that corresponding to $t = cx$, or $\theta = c$, nearly, when the integrand in the second term changes sign by passing through infinity. This is the epoch of the main shock; the minor disturbance which sets in when $t = ax$ leads up continuously to this, and only dies out gradually after it.

As a first step we may tabulate the function V (θ) defined by

$$\mathrm{V}(\theta) = -\frac{b^3(2\theta^2 - b^2)^2\sqrt{(\theta^2 - a^2)}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)}, \text{ for } a < \theta < b,$$

$$= -\frac{b^3\sqrt{(\theta^2 - a^2)}}{(2\theta^2 - b^2)^2 - 4\theta^2\sqrt{(\theta^2 - a^2)}\sqrt{(\theta^2 - b^2)}}, \text{ for } \theta > b \quad . \quad . \quad (97).$$

θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).
1·000	0	1·025	– ·39425	1·10	– ·08981	1·550	– ·22781
1·001	– ·21995	1·030	– ·36340	1·15	– ·02454	1·600	– ·31645
1·002	– ·29488	1·035	– ·33293	1·20	– ·00218	1·625	– ·37299
1·003	– ·34284	1·040	– ·30387	1·25	– ·00193	1·650	– ·44110
1·004	– ·37630	1·050	– ·25142	1·30	– ·01508	1·675	– ·52493
1·005	– ·40039	1·060	– ·20681	1·35	– ·03796	1·700	– ·63087
1·010	– ·44907	1·070	– ·16932	1·40	– ·06941	1·725	– ·76935
1·015	– ·44543	1·080	– ·13795	1·45	– ·10989	b/a	– ·81649
1·020	– ·42324	1·090	– ·11173	1·50	– ·16137	—	—

θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).	θ/a.	V (θ).
b/a	– 0·81649	1·90	+ 20·38685	2·10	+ 1·99591	2·5	+ ·91464
1·75	– 1·39031	1·95	+ 5·42335	2·15	+ 1·69743	3·0	+ ·60196
1·80	– 2·98197	2·00	+ 3·31759	2·20	+ 1·48891	4·0	+ ·38179
1·85	– 8·65843	2·05	+ 2·46398	2·25	+ 1·33404	10·0	+ ·13292
c/a	∞	—	—	—	—	—	—

The function has a minimum value $-$ ·45120 when $\theta/a = 1{\cdot}01170$, and a zero maximum when $\theta/a = 1{\cdot}22474$; it changes from $-\infty$ to $+\infty$ when $\theta/b = 1{\cdot}08767$, or $\theta/a = 1{\cdot}88389$.* Its graph is shown in the lower part of fig. 4, and also (on a smaller scale, so as to bring in a greater range of θ) in fig. 5.

It is postulated that the function Q (t) is sensible only for values of t lying within a short range on each side of 0; the function Q $(t - \theta x)$ will therefore be sensible only for values of θ in the neighbourhood of t/x. We will suppose that for given values of x and t its graph (as a function of θ) has some such form as that of the dotted curve in fig. 5. If x be constant, the effect of increasing t will be to cause this graph to travel uniformly from left to right; and if we imagine that in each of its positions the integral of the product of the ordinates of the two curves is taken, we get a mental picture of the variation of v_0 as a function of t, on a certain scale.

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* As in the case of U (θ), the calculations are due chiefly to Mr. WOODALL.

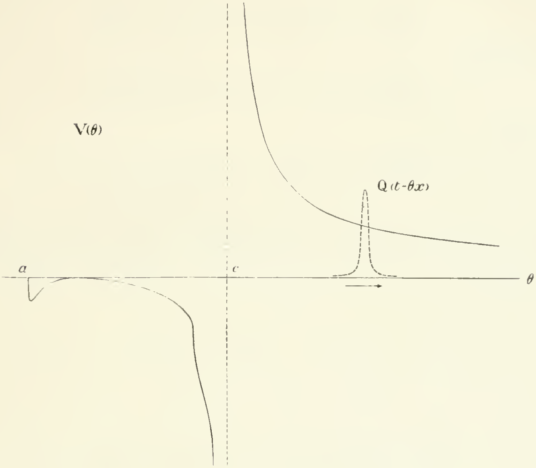

Fig. 5.

For the greater part of the range of t, the integral will be approximately proportional to the ordinates of the curve $V(\theta)$, viz., we shall have

$$v_0 = \frac{Q}{\pi\mu b x} \cdot V\left(\frac{t}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98),$$

in analogy with (95). But for a short range of t, in the neighbourhood of cx, the statement must be modified, the dotted curve being then in the neighbourhood of the vertical asymptote of the function $V(\theta)$. Since the *principal value* of the integral is to be taken, it is evident that as t approaches the critical epoch and passes it, v_0 will sink to a relatively low minimum, and then passing through zero will attain a correspondingly high maximum, after which it will decrease asymptotically to zero, the later stages coming again under the formula (98).

Although the above argument gives perhaps the best view of the whole course of the disturbance, we are not dependent upon it for a knowledge of what takes place about the critical epoch cx. We may proceed, instead, by generalizing the expressions (77). This introduces, in addition to the given function $Q(t)$, whose Fourier expression is

$$Q(t) = \frac{1}{\pi}\int_0^\infty dp \int_{-\infty}^{\infty} Q(\lambda)\cos p(t-\lambda)\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad (99),$$

the related function

$$Q^{\backprime}(t) = \frac{1}{\pi}\int_0^\infty dp \int_{-\infty}^{\infty} Q(\lambda)\sin p(t-\lambda)\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad (100);$$

viz., we have

$$u_0 = -\frac{H}{\mu} Q(t-cx) + \&c., \quad v_0 = \frac{K}{\mu} Q^{\backprime}(t-cx) + \&c. \quad . \quad . \quad (101).$$

It does not appear that the connection between the functions $Q(t)$ and $Q^{\backprime}(t)$ has been specially studied, although it presents itself in more than one department of mathematical physics. The following cases may be noted as of interest from our present point of view :

$$Q(t) = \frac{\overline{Q}}{\pi}\frac{\tau}{t^2+\tau^2}, \quad Q^{\backprime}(t) = \frac{\overline{Q}}{\pi}\frac{t}{t^2+\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (102);$$

$$Q(t) = \frac{Ct}{t^2+\tau^2}, \quad Q^{\backprime}(t) = -\frac{C\tau}{t^2+\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103);$$

$$\left.\begin{aligned} Q(t) &= \frac{\overline{Q}}{2\tau} \text{ for } t^2 < \tau^2, \\ &= 0 \text{ for } t^2 > \tau^2, \end{aligned}\right\} \quad Q^{\backprime}(t) = \frac{\overline{Q}}{4\pi\tau}\log\left(\frac{t+\tau}{t-\tau}\right)^2 \quad . \quad . \quad . \quad (104).$$

It is evident, generally, that if Q be an odd function, $Q^{\backprime}$ will be an even function, and *vice versâ*.

The values of u_0 and v_0, as given by (101), are represented graphically in fig. 6, for the case where $Q(t)$ and $Q^{\backprime}(t)$ have the forms given in (102).* Moreover, writing

$$H\overline{Q}/2\pi\mu\tau = f, \quad K\overline{Q}/2\pi\mu\tau = g, \quad t - cx = \tau\tan\chi,$$

we have

$$u_0 = -(1+\cos 2\chi)\,.f, \quad v_0 = \sin 2\chi\,.g \quad . \quad . \quad . \quad . \quad . \quad (105);$$

the orbit of a surface-particle is therefore an ellipse with horizontal and vertical semi-axes f and g. And if from the equilibrium position O we project any other position P of the particle on to a vertical straight line, the law of P's motion is that the projection (R) describes this line with constant velocity. See fig. 7, where the positive direction of y is supposed to be downwards.

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* The relation between the scales of the ordinates in the graphs of u_0 and v_0 depends upon the ratio of the elastic constants λ, μ. The figures are constructed on the hypothesis of $\lambda = \mu$.

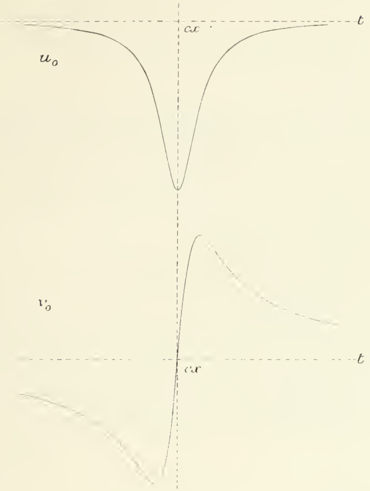

Fig. 6.

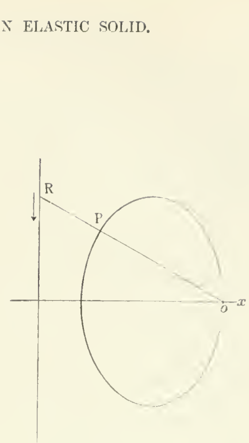

Fig. 7.

A similar treatment would apply to the formulæ (81), and (with some modification) to (84).

It remains to justify these approximations by showing that the residual disturbance tends with increasing x to the limit 0. For this purpose we have recourse to the formulæ of Art. 8. As a sufficient example, take the second term in the last member of (88). If we multiply by e^{ipt}, take the real part, and substitute $\eta = p\phi$, $k = pb$, the corresponding term in the value of v_0, as given by (52), assumes the form*

$$\frac{Q}{\mu} \cos p (t - bx) \int_0^\infty F(\phi) e^{-px\phi} d\phi + \frac{Q}{\mu} \sin p (t - bx) \int_0^\infty f(\phi) e^{-px\phi} d\phi,$$

where the functions $F(\phi)$ and $f(\phi)$, which do not involve p, are of the order ϕ^{-1} when ϕ is large. If we generalize this expression by FOURIER's Theorem (see equation (99)), we obtain, in the case of an impulse $\bar{Q}$ of short duration,

$$\frac{\bar{Q}}{\pi\mu} \int_0^\infty F(\phi)\, d\phi \int_0^\infty e^{-x\phi p} \cos p (t - bx)\, dp + \frac{\bar{Q}}{\pi\mu} \int_0^\infty f(\phi)\, d\phi \int_0^\infty e^{-x\phi p} \sin p (t - bx)\, dp$$

$$= \frac{\bar{Q}}{\pi\mu} \int_0^\infty F(\phi) \frac{x\phi\, d\phi}{x^2\phi^2 + (t - bx)^2} + \frac{\bar{Q}}{\pi\mu} \int_0^\infty f(\phi) \frac{(t - bx)\, d\phi}{x^2\phi^2 + (t - bx)^2} \quad . \quad . \quad (106).$$

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* The symbols ϕ, F, f are here used temporarily in new senses.

For any particular phase of the motion, t varies as x, and the expression (106) therefore varies inversely as x. This confirms, so far, our previous results (95) and (98). Hence with increasing distance from the origin the disturbance tends to the limiting form represented by (101).

Before leaving this part of the subject, it is to be remarked that the peculiar protracted character of the minor tremor which we have found to precede and follow the main shock is to some extent special to the two-dimensional form of the question. It is connected with the fact, dwelt upon by the author in a recent paper,* that even in an unlimited medium a solitary cylindrical wave, whether of the irrotational or equivoluminal kind, is not sharply defined in the rear, as it is in front, but is prolonged in the form of a "tail." In the three-dimensional problems, to which we are about to proceed, this cause operates in another way. The internal waves are now spherical instead of cylindrical, and so far there is no reason to expect a protraction of a disturbance which in its origin was of finite duration. But at the surface they manifest themselves as annular waves, and accordingly we shall find clear indications of the peculiarity of two-dimensional propagation to which reference has been made. On the whole, however, it appears that the epochs of arrival of irrotational and equivoluminal waves are relatively more clearly marked and isolated than in the two-dimensional cases.

PART II.

THREE-DIMENSIONAL PROBLEMS.

10. Assuming symmetry about the axis of z, we write

$$\varpi = \sqrt{(x^2 + y^2)}, \qquad u = \frac{x}{\varpi}\, q, \qquad v = \frac{y}{\varpi}\, q \quad . \quad . \quad . \quad . \quad (107),$$

so that q denotes displacement perpendicular to that axis.

A typical solution of the elastic equations, convenient for our purposes, is derived at once from Art. 3, if we imagine an infinite number of two-dimensional vibration-types of the kind specified by (25) and (30) to be arranged uniformly in all azimuths about the axis of z, and take the mean. In this way we obtain from (33), with the necessary change of notation,

$$\left.\begin{aligned} q_0 &= (i\xi\mathrm{A} - \beta\mathrm{B}) \,.\, \frac{1}{\pi}\int_0^{\pi} e^{i\xi\varpi\cos\omega}\cos\omega\, d\omega = -(\xi\mathrm{A} + i\beta\mathrm{B})\,\mathrm{J}_1(\xi\varpi) \\ w_0 &= (-\alpha\mathrm{A} - i\xi\mathrm{B}) \,.\, \frac{1}{\pi}\int_0^{\pi} e^{i\xi\varpi\cos\omega}\, d\omega = -(\alpha\mathrm{A} + i\xi\mathrm{B})\,\mathrm{J}_0(\xi\varpi) \end{aligned}\right\} . \quad (108).$$

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* Cited on p. 37 *post*.

Also, from (40), for the corresponding stresses at the plane $z = 0$, we have

$$\left.\begin{aligned} [p_{z\varpi}]_0 &= \mu\,\{2\xi\alpha \mathrm{A} + i\,(2\xi^2 - k^2)\,\mathrm{B}\}\,\mathrm{J}_1(\xi\varpi) \\ [p_{zz}]_0 &= \mu\,\{(2\xi^2 - k^2)\,\mathrm{A} + 2i\xi\beta\mathrm{B}\}\,\mathrm{J}_0(\xi\varpi) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (109).$$

Although the above derivation is sufficient for our purpose, it may be worth while to give the direct investigation,* starting from the equations

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u, \qquad \rho\frac{\partial^2 v}{\partial t^2} = (\lambda + \mu)\frac{\partial\Delta}{\partial y} + \mu\nabla^2 v,$$

$$\rho\frac{\partial^2 w}{\partial t^2} = (\lambda + \mu)\frac{\partial\Delta}{\partial z} + \mu\nabla^2 w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110),$$

where

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111).$$

In the case of simple-harmonic motion (e^{ipt}) these are satisfied by

$$u = \frac{\partial\phi}{\partial x} + u', \qquad v = \frac{\partial\phi}{\partial y} + v', \qquad w = \frac{\partial\phi}{\partial z} + w' \quad . \quad . \quad . \quad . \quad (112),$$

provided

$$(\nabla^2 + h^2)\,\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113),$$

and

$$\left.\begin{aligned} (\nabla^2 + k^2)\,u' = 0, \quad (\nabla^2 + k^2)\,v' = 0, \quad (\nabla^2 + k^2)\,w' = 0 \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (114).$$

where h^2, k^2 are defined as before by (28). A particular solution of (114) is

$$u' = \frac{\partial^2\chi}{\partial x\,\partial z}, \quad v' = \frac{\partial^2\chi}{\partial y\,\partial z}, \quad w' = \frac{\partial^2\chi}{\partial z^2} + k^2\chi \quad . \quad . \quad . \quad . \quad (115).$$

provided

$$(\nabla^2 + k^2)\,\chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

On the hypothesis of symmetry about Oz we have

$$\nabla^2 = \frac{\partial^2}{\partial\varpi^2} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi} + \frac{\partial^2}{\partial z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117),$$

and the formulæ (112), (115) are equivalent to

$$q = \frac{\partial\phi}{\partial\varpi} + \frac{\partial^2\chi}{\partial\varpi\,\partial z}, \quad w = \frac{\partial\phi}{\partial z} + \frac{\partial^2\chi}{\partial z^2} + k^2\chi \quad . \quad . \quad . \quad . \quad (118).$$

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* *Cf.* 'Proc. Lond. Math. Soc.,' vol. 34, p. 276, for the corresponding statical investigation.

If we take, as the typical solution of (113) and (116),

$$\phi = \mathrm{A}e^{-\alpha z}\,\mathrm{J}_0(\xi\varpi), \quad \chi = \mathrm{B}e^{-\beta z}\,\mathrm{J}_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \qquad (119),$$

where α, β have the same meanings and are subject to the same convention as in Art. 3, we have, from (118),

$$\left.\begin{aligned} q &= (-\,\xi\mathrm{A}e^{-\alpha z} + \xi\beta\mathrm{B}e^{-\beta z})\,\mathrm{J}_1(\xi\varpi) \\ w &= (-\,\alpha\mathrm{A}e^{-\alpha z} + \xi^2\mathrm{B}e^{-\beta z})\,\mathrm{J}_0(\xi\varpi) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (120);$$

and thence for the stresses in the plane $z = 0$

$$\left.\begin{aligned} [p_{z\varpi}]_0 &= \mu\left[\frac{\partial q}{\partial z} + \frac{\partial w}{\partial \varpi}\right]_0 = \mu\,\{2\xi\alpha\mathrm{A} - (2\xi^2 - k^2)\,\xi\mathrm{B}\}\,\mathrm{J}_1(\xi\varpi) \\ [p_{zz}]_0 &= \left[\lambda\Delta + 2\mu\,\frac{\partial w}{\partial z}\right]_0 = \mu\,\{(2\xi^2 - k^2)\,\mathrm{A} - 2\xi^2\beta\mathrm{B}\}\,\mathrm{J}_0(\xi\varpi) \end{aligned}\right\} \quad . \qquad (121).$$

The formulæ differ from (108) and (109) only in the substitution of $i\xi\mathrm{B}$ for B. The notation of (119) is adopted as the basis of the subsequent calculations.

If we are to assume, in place of (119),

$$\phi = \mathrm{A}'e^{\alpha z}\mathrm{J}_0(\xi\varpi), \quad \chi = \mathrm{B}'e^{\beta z}\mathrm{J}_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (122),$$

the corresponding forms of (120) and (121) would be obtained by affixing accents to A and B, and changing the signs of α and β where they occur explicitly.

11. As in Art. 4, we begin by applying the preceding formulæ to the solution of a known problem, viz., where a given periodic force acts at a point in an unlimited solid.

Let us suppose, in the first place, that an extraneous force of amount $\mathrm{Z}\,.\,\mathrm{J}_0(\xi\varpi)\,e^{ipt}$, per unit area, acts parallel to z on an infinitely thin stratum coincident with the plane $z = 0$. The formulæ (119) will then apply for $z > 0$, and (122) for $z < 0$. The normal stress will be discontinuous, viz. :

$$[p_{zz}]_{z=+0} - [p_{zz}]_{z=-0} = -\,\mathrm{Z}\,.\,\mathrm{J}_0(\xi\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (123),$$

whilst $p_{z\varpi}$ is continuous. Hence

$$\left.\begin{aligned} (2\xi^2 - k^2)(\mathrm{A} - \mathrm{A}') - 2\xi^2\beta\,(\mathrm{B} + \mathrm{B}') &= -\,\frac{\mathrm{Z}}{\mu} \\ 2\alpha\,(\mathrm{A} + \mathrm{A}') - (2\xi^2 - k^2)(\mathrm{B} - \mathrm{B}') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \qquad (124).$$

Also, the continuity of q and ϖ requires

$$\left.\begin{aligned} \mathrm{A} - \mathrm{A}' - \beta\,(\mathrm{B} + \mathrm{B}') &= 0 \\ \alpha\,(\mathrm{A} + \mathrm{A}') - \xi^2\,(\mathrm{B} - \mathrm{B}') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (125).$$

We infer

$$\mathrm{A} = -\,\mathrm{A}' = \frac{\mathrm{Z}}{2k^2\mu}, \quad \mathrm{B} = \mathrm{B}' = \frac{\mathrm{Z}}{2k^2\mu\beta} \quad . \quad . \quad . \quad . \quad . \qquad (126),$$

and therefore, for $z > 0$,

$$\phi = \frac{Z}{2k^2\mu} e^{-\alpha z} J_0(\xi\varpi), \quad \chi = \frac{Z}{2k^2\mu} \frac{e^{-\beta z}}{\beta} J_0(\xi\varpi) . \quad . \quad . \quad . \quad . \quad (127).$$

To pass to the case of a concentrated force Re^{ipt}, acting parallel to z at the origin, we have recourse to the formula (20), where we suppose $f(\lambda)$ to vanish for all but infinitesimal values of λ, and to become infinite for these in such a way that

$$\int_0^\infty f(\lambda)\, 2\pi\lambda\, d\lambda = R.$$

We therefore write $Z = R\xi\, d\xi/2\pi$, and integrate with respect to ξ from 0 to ∞.[*] We thus find, for $z > 0$,

$$\phi = \frac{R}{4\pi p^2\rho} \int_0^\infty e^{-\alpha z} J_0(\xi\varpi)\, \xi\, d\xi, \qquad \chi = \frac{R}{4\pi p^2\rho} \int_0^\infty \frac{e^{-\beta z}}{\beta} J_0(\xi\varpi)\, \xi\, d\xi . \quad . \quad (128),$$

which are equivalent, by (18), to

$$\phi = -\frac{R}{4\pi p^2\rho} \cdot \frac{\partial}{\partial z} \frac{e^{-ihr}}{r}, \qquad \chi = \frac{R}{4\pi p^2\rho} \cdot \frac{e^{-ikr}}{r} \quad . \quad . \quad . \quad . \quad (129).$$

This will be found to agree with the known solution of the problem.[†] If we retain only the terms which are most important at a great distance r, we find, from (118),

$$\left. \begin{aligned} q &= \frac{R}{4\pi} \left\{ \frac{1}{\lambda + 2\mu} \frac{z\varpi}{r^3} e^{-ihr} - \frac{1}{\mu} \frac{z\varpi}{r^3} e^{-ikr} \right\} \\ w &= \frac{R}{4\pi} \left\{ \frac{1}{\lambda + 2\mu} \frac{z^2}{r^3} e^{-ihr} + \frac{1}{\mu} \frac{\varpi^2}{r^3} e^{-ikr} \right\} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (130).$$

Inserting the time-factor, the radial displacement is

$$\frac{zw + \varpi q}{r} = \frac{R}{4\pi(\lambda + 2\mu)} \cdot \frac{z}{r^2} e^{ip(t-ar)} \quad . \quad . \quad . \quad . \quad . \quad (131),$$

and the transverse displacement in the meridian plane is

$$\frac{\varpi w - zq}{r} = \frac{R}{4\pi\mu} \cdot \frac{\varpi}{r^2} e^{ip(t-br)} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132).$$

Returning to the exact formulæ (128), the expression for the velocity parallel to z at the plane $z = 0$ is found to be

$$\frac{\partial w}{\partial t} = \frac{iRe^{ipt}}{4\pi p\rho} \int_0^\infty \left(-\alpha + \frac{\xi^2}{\beta} \right) J_0(\xi\varpi)\, \xi\, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad (133),$$

---

[*] A more rigorous procedure would be to suppose in the first instance that the force R is uniformly distributed over a circular area of radius a, using the formula (22). If in the end we make $a = 0$, we obtain the results in the text.

[†] STOKES, 'Camb. Trans.,' vol. 9 (1849); 'Mathematical and Physical Papers,' vol. 2, p. 278.

or, taking the real part,

$$\frac{\partial w}{\partial t} = \frac{\mathrm{R}}{4\pi p\rho}\left\{\int_0^k \frac{\xi^3}{\sqrt{(k^2-\xi^2)}}\,\mathrm{J}_0(\xi\varpi)\,d\xi + \int_0^h \xi\sqrt{(h^2-\xi^2)}\,\mathrm{J}_0(\xi\varpi)\,d\xi\right\}\cos pt$$
$$+ \text{ terms in } \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134).$$

The terms in cos pt remain finite when we put $\varpi = 0$;* and the mean rate W at which a force R cos pt does work in generating waves is thus found to be

$$\mathrm{W} = \frac{\mathrm{R}^2}{8\pi p\rho}\left\{\int_0^k \frac{\xi^3\,d\xi}{\sqrt{(k^2-\xi^2)}} + \int_0^h \xi\sqrt{(h^2-\xi^2)}\,d\xi\right\}$$
$$= \frac{\mathrm{R}^2}{24\pi p\rho}\,.\,(2k^3+h^3) = \frac{p^2\mathrm{R}^2}{24\pi\rho}(a^3+2b^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (135),$$

a and b denoting as before the two elastic wave-slownesses. The result (135) can be deduced, as a particular case, from formulæ given by Lord KELVIN.†

12. Proceeding to the case of a semi-infinite solid occupying (say) the region $z > 0$, we begin with the special distribution of surface-stress:

$$[p_{zz}]_0 = \mathrm{Z}\,.\,\mathrm{J}_0(\xi\varpi), \quad [p_{z\varpi}] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136).$$

The coefficients A, B in (119) are now determined by

$$\left.\begin{aligned}(2\xi^2-k^2)\,\mathrm{A} - 2\xi^2\beta\mathrm{B} &= \frac{\mathrm{Z}}{\mu}\\ 2\alpha\mathrm{A} - (2\xi^2-k^2)\,\mathrm{B} &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137),$$

whence

$$\mathrm{A} = \frac{2\xi^2-k^2}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Z}}{\mu}, \quad \mathrm{B} = \frac{2\alpha}{\mathrm{F}(\xi)}\cdot\frac{\mathrm{Z}}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138),$$

the function F (ξ) having the same meaning as in Art. 5. The corresponding surface-displacements are

$$\left.\begin{aligned}q_0 &= -\frac{\xi(2\xi^2-k^2-2\alpha\beta)}{\mathrm{F}(\xi)}\cdot\mathrm{J}_1(\xi\varpi)\cdot\frac{\mathrm{Z}}{\mu}\\ w_0 &= \frac{k^2\alpha}{\mathrm{F}(\xi)}\cdot\mathrm{J}_0(\xi\varpi)\cdot\frac{\mathrm{Z}}{\mu}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (139).$$

This result might have been deduced immediately from (51) in the manner indicated at the beginning of Art. 10.

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* The terms in sin pt become infinite. If the force R be distributed over a circular area, the awkwardness is avoided. A factor

$$\left\{\frac{\mathrm{J}_1(\xi a)}{\frac{1}{2}\xi a}\right\}^2$$

is thus introduced under the integral signs in the first line of (135), where a denotes (for the moment) the radius of the circle. Finally, we can make a infinitely small.

† 'Phil. Mag.,' Aug. 1899, pp. 234, 235.

If we put Z = 0 in (137) we get a system of free annular surface-waves, in which

$$\left.\begin{aligned} q_0 &= -\kappa(2\kappa^2 - k^2 - 2\alpha_1\beta_1) \,.\, J_1(\kappa\varpi) \,.\, Ce^{ipt} \\ w_0 &= k^2\alpha_1 \,.\, J_0(\kappa\varpi) \,.\, Ce^{ipt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (140),$$

where κ is the positive root of $F(\xi) = 0$, and α_1, β_1 are the corresponding values of α, β. These are of the nature of "standing" waves.

To pass to the case of a concentrated vertical pressure Re^{ipt} at O.* we put in accordance with (20), $Z = -R\xi d\xi/2\pi$, and integrate from 0 to ∞.† The formulæ (139) become

$$\left.\begin{aligned} q_0 &= \frac{R}{2\pi\mu}\int_0^\infty \frac{\xi^2(2\xi^2 - k^2 - 2\alpha\beta)}{F(\xi)} J_1(\xi\varpi)\,d\xi \\ w_0 &= -\frac{R}{2\pi\mu}\int_0^\infty \frac{k^2\xi\alpha}{F(\xi)} J_0(\xi\varpi)\,d\xi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (141).$$

Again, the case of an internal source of the type

$$\phi = \frac{e^{-ihr}}{r}, \quad \chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142),$$

where r denotes distance from the point $(0, 0, f)$, can be solved by a process similar to that of Art. 5. First, superposing an equal source at $(0, 0, -f)$, distance from which is denoted by r', we have

$$\phi = \frac{e^{-ihr}}{r} + \frac{e^{-ihr'}}{r'}, \quad \chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143);$$

and therefore, by (18), in the neighbourhood of the plane $z = 0$,

$$\begin{aligned} \phi &= \int_0^\infty \frac{e^{-\alpha(z+f)}}{\alpha} J_0(\xi\varpi)\,\xi d\xi + \int_0^\infty \frac{e^{\alpha(z-f)}}{\alpha} J_0(\xi\varpi)\,\xi d\xi \\ &= 2\int_0^\infty \frac{\cosh\alpha z}{\alpha} e^{-\alpha f} J_0(\xi\varpi)\,\xi d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144). \end{aligned}$$

This makes

$$q_0 = -2\int_0^\infty \frac{e^{-\alpha f}}{\alpha} J_1(\xi\varpi)\,\xi^2\,d\xi, \quad w_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (145).$$

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* This may be regarded as the kinetic analogue of BOUSSINESQ's well-known statical problem.

† It might appear at first sight that a simpler procedure would be possible, and that the effect of a pressure concentrated at a point might be inferred by superposing *lines* of pressure (through O) uniformly in all azimuths, and using the results of § 7. It is easily seen, however, that such a distribution of lines of pressure is equivalent to a pressure-intensity varying inversely as the distance (ϖ) from O. This is not an adequate representation of a localized pressure, since it makes the total pressure on a circular area having its centre at O increase indefinitely with the radius of the circle.

and

$$[p_{z\varpi}]_0 = 0, \quad [p_{zz}]_0 = 2\mu \int_0^\infty \frac{(2\xi^2 - k^2)}{\alpha} e^{-\alpha z} \mathrm{J}_0(\xi\varpi)\, \xi\, d\xi \, . \quad . \quad . \quad (146).$$

The additions to (143) which are required in order to annul the stresses on the plane $z = 0$ are accordingly found by writing

$$\mathrm{Z} = -2\mu \cdot \frac{2\xi^2 - k^2}{\alpha} e^{-\alpha z}\, \xi\, d\xi$$

in (139), and then integrating with respect to ξ from 0 to ∞. In this way we obtain, finally,

$$\left.\begin{aligned} q_0 &= 4 \int_0^\infty \frac{k^2\xi^2\beta}{\mathrm{F}(\xi)} e^{-\alpha z} \mathrm{J}_1(\xi\varpi)\, d\xi \\ w_0 &= -2 \int_0^\infty \frac{k^2\xi(2\xi^2 - k^2)}{\mathrm{F}(\xi)} e^{-\alpha z} \mathrm{J}_0(\xi\varpi)\, d\xi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (147).$$

In a similar manner, with the help of Art. 11, we might calculate the effect of a periodic vertical force, acting at an internal point.

13. For the sake of comparison with our previous two-dimensional formulæ, it is convenient to write, from (2) and (6),

$$\left.\begin{aligned} \mathrm{J}_0(\xi\varpi) &= -\frac{i}{\pi} \int_0^\infty (e^{i\xi\varpi\cosh u} - e^{-i\xi\varpi\cosh u})\, du \\ \mathrm{J}_1(\xi\varpi) &= -\frac{1}{\pi} \int_0^\infty (e^{i\xi\varpi\cosh u} + e^{-i\xi\varpi\cosh u}) \cosh u\, du \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (148).$$

The formulæ (141) are thus equivalent to

$$\left.\begin{aligned} q_0 &= -\frac{\mathrm{R}}{2\pi^2\mu} \int_0^\infty \cosh u\, du \int_{-\infty}^\infty \frac{\xi^2(2\xi^2 - k^2 - 2\alpha\beta)}{\mathrm{F}(\xi)} e^{i\xi\varpi\cosh u}\, d\xi \\ w_0 &= \frac{i\mathrm{R}}{2\pi^2\mu} \int_0^\infty du \int_{-\infty}^\infty \frac{k^2\xi\alpha}{\mathrm{F}(\xi)} e^{i\xi\varpi\cosh u}\, d\xi \end{aligned}\right\} . \quad . \quad (149).$$

These results are closely comparable with (52), and our previous methods of treatment will apply. It is, however, unnecessary to go through all the details of the work, since the definite integrals with respect to ξ which appear in (149) can be derived from those in (52) by performing the operation $-i\partial/\partial x$ upon the latter, and then replacing x by $\varpi \cosh u$.

Thus, from (67) and (70) we derive

$$\begin{aligned} \mathfrak{P}\int_{-\infty}^\infty \frac{\xi^2(2\xi^2 - k^2 - 2\alpha\beta)}{\mathrm{F}(\xi)} e^{i\xi\varpi\cosh u}\, d\xi &= 2\pi\kappa\mathrm{H} \sin(\kappa\varpi\cosh u) \\ &\quad + 4k^2 \int_h^k \frac{\xi^2(2\xi^2 - k^2)\,\alpha\beta}{\mathrm{F}(\xi)\, f(\xi)} e^{-i\xi\varpi\cosh u}\, d\xi \quad . \quad (150), \end{aligned}$$

$$\mathfrak{P}\int_{-\infty}^{\infty}\frac{k^2\xi\alpha}{\mathrm{F}(\xi)}e^{i\xi\varpi\cosh u}\,d\xi = 2\pi\kappa\mathrm{K}\sin(\kappa\varpi\cosh u) - 2k^2\mathfrak{P}\int_{k}^{\infty}\frac{\xi\alpha}{\mathrm{F}(\xi)}e^{-i\xi\varpi\cosh u}\,d\xi$$
$$- 2k^2\int_{h}^{k}\frac{\xi(2\xi^2-k^2)^2\alpha}{\mathrm{F}(\xi)f(\xi)}e^{-i\xi\varpi\cosh u}\,d\xi \quad . \quad . \quad . \quad . \quad . \quad (151),$$

where H and K are the numerical quantities defined by (68) and (71). Substituting in (149) we have

$$\mathfrak{P}q_0 = -\frac{\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{H}\,.\,\mathrm{K}_1(\kappa\varpi) + \frac{ik^2\mathrm{R}}{\pi\mu}\int_{h}^{k}\frac{\xi^2(2\xi^2-k^2)\alpha\beta}{\mathrm{F}(\xi)f(\xi)}\mathrm{D}_1(\xi\varpi)\,d\xi \quad . \quad (152),$$

$$\mathfrak{P}w_0 = \frac{i\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{K}\,.\,\mathrm{J}_0(\kappa\varpi) - \frac{ik^2\mathrm{R}}{2\pi\mu}\mathfrak{P}\int_{k}^{\infty}\frac{\xi\alpha}{\mathrm{F}(\xi)}\mathrm{D}_0(\xi\varpi)\,d\xi$$
$$- \frac{ik^2\mathrm{R}}{2\pi\mu}\int_{h}^{k}\frac{\xi(2\xi^2-k^2)^2\alpha}{\mathrm{F}(\xi)f(\xi)}\mathrm{D}_0(\xi\varpi)\,d\xi \quad . \quad . \quad (153),$$

where the notation of the various BESSEL's Functions is as in Art. 2.

Superposing the system of free waves in which

$$q_0 = \frac{i\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{H}\,.\,\mathrm{J}_1(\kappa\varpi), \quad w_0 = -\frac{i\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{K}\,.\,\mathrm{J}_0(\kappa\varpi) \quad . \quad . \quad . \quad . \quad (154),$$

we obtain, finally, on inserting the time-factor,

$$q_0 = -\frac{\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{H}\,.\,\mathrm{D}_1(\kappa\varpi)\,e^{ipt} + \frac{ik^2\mathrm{R}}{\pi\mu}\int_{h}^{k}\frac{\xi^2(2\xi^2-k^2)\alpha\beta}{\mathrm{F}(\xi)f(\xi)}\mathrm{D}_1(\xi\varpi)\,e^{ipt}\,d\xi \quad . \quad . \quad (155),$$

$$w_0 = -\frac{ik^2\mathrm{R}}{2\pi\mu}\mathfrak{P}\int_{k}^{\infty}\frac{\xi\alpha}{\mathrm{F}(\xi)}\mathrm{D}_0(\xi\varpi)\,e^{ipt}\,d\xi - \frac{ik^2\mathrm{R}}{2\pi\mu}\int_{h}^{k}\frac{\xi(2\xi^2-k^2)^2\alpha}{\mathrm{F}(\xi)f(\xi)}\mathrm{D}_0(\xi\varpi)\,e^{ipt}\,d\xi \quad . \quad (156).$$

Since these expressions are made up entirely of diverging waves, they constitute the complete solution of the problem where a periodic normal force $\mathrm{R}e^{ipt}$ is applied to the surface at the origin.

An alternative form of (156), which puts in evidence that part of the vertical disturbance which is most important at a great distance from the origin, is obtained from (75). Attending only to the "singular" term, we find

$$\mathfrak{P}\int_{-\infty}^{\infty}\frac{k^2\xi\alpha}{\mathrm{F}(\xi)}e^{i\xi\varpi\cosh u}\,d\xi = -2i\pi\kappa\mathrm{K}\,.\,\cos(\kappa\varpi\cosh u) + \&\text{c.} \quad . \quad . \quad . \quad (157),$$

and therefore, from (149),

$$\mathfrak{P}w_0 = \frac{\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{K}\,.\,\mathrm{K}_0(\kappa\varpi) + \&\text{c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158).$$

Adding in the system (154) we have altogether

$$q_0 = -\frac{\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{H}\,.\,\mathrm{D}_1(\kappa\varpi)\,e^{ipt} + \&\text{c.}, \quad w_0 = \frac{\kappa\mathrm{R}}{2\mu}\,.\,\mathrm{K}\,.\,\mathrm{D}_0(\kappa\varpi)\,e^{ipt} + \&\text{c.} \qquad (159).$$

Hence, by (7), we have, at a great distance ϖ,

$$q_0 = -\frac{i\kappa R}{2\mu} H . \sqrt{\frac{2}{\pi\kappa\varpi}} . e^{i(pt-\kappa\varpi-\frac{1}{4}\pi)}, \quad w_0 = \frac{\kappa R}{2\mu} K . \sqrt{\frac{2}{\pi\kappa\varpi}} . e^{i(pt-\kappa\varpi-\frac{1}{4}\pi)}. \quad (160).$$

This may be compared with (77). The vibrations are elliptic, with the same ratio of horizontal and vertical diameters as in the case of two dimensions; but the amplitude diminishes with increasing distance according to the usual law $\varpi^{-\frac{1}{2}}$ of annular divergence.

In the same manner we obtain, in the case of an internal source of the type (142),

$$\left. \begin{aligned} q_0 &= -\frac{4\pi k^2\kappa\beta_1}{F'(\kappa)} e^{-\alpha_1 f} D_1(\kappa\varpi) e^{ipt} + \&c., \\ w_0 &= \frac{2\pi k^2\kappa(2\kappa^2 - k^2)}{F'(\kappa)} e^{-\alpha_1 f} D_0(\kappa\varpi) e^{ipt} + \&c. \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (161),$$

where the factor $e^{-\alpha_1 f}$ shows the effect of the depth of the source.

The expressions for the residual disturbance might be derived from the formulæ of Art. 8 by the same artifice. Without attempting to give the complete results, which would be somewhat complicated, it may be sufficient to ascertain their general form, and order of magnitude, when $h\varpi$ and $k\varpi$ are large. To take, for example, the parts due to the distortional waves, if we perform the operation $-i\partial/\partial x$ on the second terms of the unnumbered expressions which occur between equations (89) and (90), above, and then replace x by $\varpi \cosh u$, the more important part of the result in each case is

$$e^{-ik\varpi \cosh u}/(k\varpi \cosh u)^{3/2},$$

multiplied by a constant factor. This result is to be substituted for the definite integrals with respect to ξ which occur in (149); the corresponding terms in q_0 and w_0 are therefore of the types

$$\frac{1}{(k\varpi)^{\frac{3}{2}}}\int_0^{\infty} \frac{e^{-ik\varpi \cosh u}\, du}{(\cosh u)^{\frac{1}{2}}}, \quad \text{and} \quad \frac{1}{(k\varpi)^{\frac{3}{2}}}\int_0^{\infty} \frac{e^{-ik\varpi \cosh u}\, du}{(\cosh u)^{\frac{3}{2}}},$$

respectively. By the method by which the asymptotic expansion (7) of the function $D_0(\zeta)$ is obtained, it may be shown, again, that these terms are ultimately comparable with

$$e^{ip(t-b\varpi)}/(k\varpi)^2,$$

where the time-factor has been restored. In the same way, the terms in q_0 and w_0 which correspond to the expansional waves are ultimately comparable with

$$e^{ip(t-a\varpi)}/(h\varpi)^2.$$

The attenuation with increasing distance is much more rapid than in the case of the annular Rayleigh waves, so that the latter ultimately predominate.* It is also much more rapid than in the case of elastic waves diverging from a centre in an unlimited medium, where the amplitude varies inversely as the distance.

14. The generalization of the preceding results, so as to apply to an arbitrary time-variation of the source, follows much the same course as in Art. 9. The full interpretation is however more difficult, so far at least as regards the minor tremors.

The main part of the disturbance, in the case of a local vertical pressure applied to the surface, is obtained by generalizing the formulæ (159). These may be written

$$q_0 = \frac{\mathrm{H}}{\pi}\frac{\mathrm{R}}{\mu}\frac{\partial}{\partial\varpi}\int_0^\infty e^{ip(t-c\varpi\cosh u)}\,du + \&\mathrm{c.}, \quad w_0 = -\frac{i\mathrm{K}}{\pi}\frac{\mathrm{R}c}{\mu}\frac{\partial}{\partial t}\int_0^\infty e^{ip(t-c\varpi\cosh u)}\,du + \&\mathrm{c.} \qquad (162).$$

Hence, corresponding to an arbitrary pressure $\mathrm{R}(t)$, we have

$$q_0 = \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}(t-c\varpi\cosh u)\,du + \&\mathrm{c.}, \quad w_0 = \frac{\mathrm{K}c}{\pi\mu}\frac{\partial}{\partial t}\int_0^\infty \grave{\mathrm{R}}(t-c\varpi\cosh u)\,du + \&\mathrm{c.} \quad (163),$$

where, in analogy with (100),

$$\grave{\mathrm{R}}(t) = \frac{1}{\pi}\int_0^\infty dp\int_{-\infty}^\infty \mathrm{R}(\lambda)\sin p(t-\lambda)\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad (164).$$

The character of the function of t represented by the first definite integral in (163) has been examined by the author† for various simple forms of $\mathrm{R}(t)$, and a similar treatment applies to the second integral. For example, if we take

$$\mathrm{R}(t) = \frac{\mathrm{R}}{\pi}\frac{\tau}{t^2+\tau^2}, \qquad \grave{\mathrm{R}}(t) = \frac{\mathrm{R}}{\pi}\frac{t}{t^2+\tau^2} \quad . \quad . \quad . \quad . \quad . \quad (165),$$

it is found, on putting

$$t - c\varpi = \tau\tan\chi,§$$

that for values of ϖ large compared with τ/c, and for moderate values of χ,

$$\int_0^\infty \mathrm{R}(t-c\varpi\cosh u)\,du = \frac{\mathrm{R}}{2\tau}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}\cos\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\chi\right)\sqrt{(\cos\chi)} \quad . \quad (166)‡,$$

$$\int_0^\infty \grave{\mathrm{R}}(t-c\varpi\cosh u)\,du = -\frac{\mathrm{R}}{2\tau}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}\sin\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\chi\right)\sqrt{(\cos\chi)} \,. \quad (167),$$

approximately. Substituting in (163), we have, ignoring the residual terms,

$$\left.\begin{aligned} q_0 &= -f\sin\left(\tfrac{1}{4}\pi-\tfrac{3}{2}\chi\right)\cos^{\frac{3}{2}}\chi \\ w_0 &= g\cos\left(\tfrac{1}{4}\pi-\tfrac{3}{2}\chi\right)\cos^{\frac{3}{2}}\chi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168),$$

---

* *Cf.* the footnote on p. 2 *ante*.

† "On Wave-Propagation in Two Dimensions," 'Proc. Lond. Math. Soc.,' vol. 35, p. 141 (1902).

‡ *Cf.* Equation (36) of the paper cited. It may be noticed that the functions on the right hand of (166) and (167) are interchanged, with a change of sign, when we reverse the sign of χ.

§ The symbol χ is no longer required in the sense of equations (115), &c.

where

$$f = \mathrm{H}\frac{\mathrm{R}c}{4\pi\mu\tau^2}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}, \qquad g = \mathrm{K}\frac{\mathrm{R}c}{4\pi\mu\tau^2}\sqrt{\left(\frac{2\tau}{c\varpi}\right)}.$$

The following numerical table is derived from one given on p. 155 of the paper referred to :—

$2\chi/\pi$.	$(t - c\varpi)/\tau$.	q_0/f.	w_0/g.
– ·9	– 6·314	– ·014	– ·060
– ·8	– 3·078	– ·078	– ·153
– ·7	– 1·963	– ·199	– ·233
– ·6	– 1·376	– ·365	– ·265*
– ·5	– 1·000	– ·549	– ·228
– ·4	– ·727	– ·719	– ·114
– ·3	– ·510	– ·838	+ ·066
·2	– ·325	– ·882*	+ ·287
– ·1	– ·158	– ·837	+ ·513
0	0	– ·707	+ ·707
+ ·1	+ ·158	– ·513	+ ·837
+ ·2	+ ·325	– ·287	+ ·882*
+ ·3	+ ·510	– ·066	+ ·838
+ ·4	+ ·727	+ ·114	+ ·719
+ ·5	+ 1·000	+ ·228	+ ·549
+ ·6	+ 1·376	+ ·265*	+ ·365
+ ·7	+ 1·963	+ ·233	+ ·199
+ ·8	+ 3·078	+ ·153	+ ·078
+ ·9	+ 6·314	+ ·060	+ ·014

* Extremes.

The graphs of q_0 and w_0 as functions of t, in the neighbourhood of the critical epoch $c\varpi$, are shown in fig. 8, which may be compared with fig. 6.† The corresponding orbit of a surface particle is traced in fig. 9, where the positive direction of z is downwards ; it may be derived by a homogeneous strain from a portion of the curve whose polar equation is

$$r^{\frac{4}{3}} = a^{\frac{4}{3}} \cos \tfrac{2}{3}\left(\theta - \tfrac{3}{4}\pi\right).$$

The amplitude of this part of the disturbance diminishes, with increasing distance from the source, according to the law $\varpi^{-\frac{1}{2}}$.

Complete expressions for the disturbance are obtained by generalizing (155) and (156). They may be written

$$q_0 = \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^{\infty}\mathrm{R}\left(t - c\varpi\cosh u\right)du - \frac{2}{\pi^2 b\mu}\int_a^b \mathrm{U}(\theta)\,.\,\frac{\partial}{\partial\varpi}\int_0^{\infty}\mathrm{R}\left(t - \theta\varpi\cosh u\right)du\,.\,d\theta \quad (169),$$

$$w_0 = \frac{1}{\pi^2 b\mu}\,\mathfrak{P}\int_a^{\infty}\theta\mathrm{V}(\theta)\,.\,\frac{\partial}{\partial t}\int_0^{\infty}\mathrm{R}\left(t - \theta\varpi\cosh u\right)du\,.\,d\theta \quad .\ .\ .\ (170),$$

where U (θ) and V (θ) are the functions defined and tabulated in Art. 9.

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† See the footnote on p. 26 *ante*.

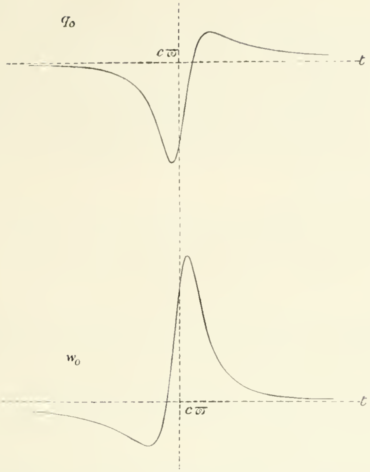

Fig. 8.

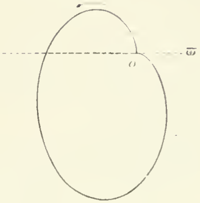

Fig. 9.

The method applied in that Article to obtain a general view of the whole progress of the vertical displacement at any point might be employed again here, the upper and lower curves in fig. 4 being combined with auxiliary movable graphs of

$$-\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}\,(t-\theta\varpi\cosh u)\,du \quad \text{and} \quad \theta\frac{\partial}{\partial t}\int_0^\infty \mathrm{R}\,(t-\theta\varpi\cosh u)\,du,$$

considered as functions of θ. In the case of a primary impulse of the type (165), both graphs would have somewhat the form of the *lower* curve in fig. 8, the functions being practically (except for a constant factor) of the type

$$\frac{\theta}{\sqrt{\varpi}}\sin\left(\tfrac{1}{4}\pi-\tfrac{3}{2}\chi\right)\cos^{\frac{3}{2}}\chi, \qquad \text{where} \qquad \chi=\tan^{-1}\frac{t-\theta\varpi}{\tau},$$

in the more important part of the range. Both graphs, if drawn to the scale of fig. 4 or 5, would be excessively contracted horizontally when we are concerned with values of ϖ large compared with τ/c_0. Owing to the compensation between positive and negative ordinates in the auxiliary graphs, it is plain that the disturbance expressed by the θ-integrals in (169) and (170) will be relatively very small except when t/ϖ has values θ for which the gradient of $\mathrm{U}(\theta)$ or $\mathrm{V}(\theta)$ is considerable. As regards the horizontal displacement q_0, the minor tremor will consist of a single to-and-fro oscillation about the epoch $a\varpi$, followed after an interval by a somewhat similar oscillation about the epoch $b\varpi$, with almost complete quiescence between. As regards the vertical displacement, there will be a to-and-fro oscillation about the epoch $a\varpi$, then a period of comparative quiescence, and finally a gradually increasing negative displacement (with a slight irregularity at the epoch $b\varpi$) leading up to the main shock, after which there is a gradually decreasing positive displacement.

The expression for the horizontal displacement q_0 may be treated in a different manner. Transforming (169) we have

$$q_0 = \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}\,(t - c\varpi\cosh u)\,du - \frac{2}{\pi^2 b\mu\varpi}\int_a^b \theta\mathrm{U}\,(\theta)\,.\,\frac{\partial}{\partial\theta}\int_0^\infty \mathrm{R}\,(t - \theta\varpi\cosh u)\,du\,.\,d\theta$$

$$= \frac{\mathrm{H}}{\pi\mu}\frac{\partial}{\partial\varpi}\int_0^\infty \mathrm{R}\,(t - c\varpi\cosh u)\,du$$

$$+ \frac{2}{\pi^2 b\mu\varpi}\int_a^b \{\theta\mathrm{U}'\,(\theta) + \mathrm{U}\,(\theta)\}\,.\int_0^\infty \mathrm{R}\,(t - \theta\varpi\cosh u)\,du\,.\,d\theta\ . \quad (171).$$

A rough sketch of the graph of $\theta\mathrm{U}'\,(\theta) + \mathrm{U}\,(\theta)$ is easily made, and the function

$$\int_0^\infty \mathrm{R}\,(t - \theta\varpi\cosh u)\,du$$

is, in such a case as (165), one-signed, but its integral with respect to θ does not converge when the lower limit is large and negative. The method therefore fails to

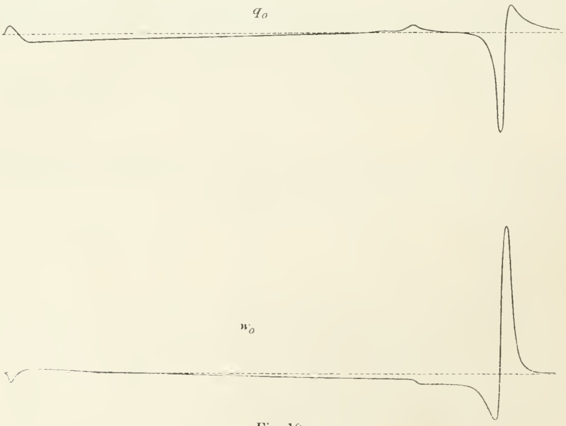

Fig. 10.

give us a convenient view of the progress of q_0 as a function of t. The difficulty is due to the peculiarities of annular propagation to which reference has already been made.

In fig. 10 an attempt, based on the former method, is made to represent (very roughly) the whole progress of the horizontal and vertical displacements due to a single impulse of the type (165) at a distance large compared with τ/c.

SUMMARY.

We may now briefly review the principal results of the foregoing investigation, so far as they may be expected to throw light on the propagation of seismic tremors over the surface of the earth.

It has been necessary to idealize this problem in various ways in order to render it amenable to calculation. In the first place, the material is taken to be compact and homogeneous, to have, in fact, the properties of the "isotropic elastic solid" of theory. Moreover, the curvature of the surface is neglected. Again, instead of a disturbance originating at an internal point, we study chiefly the case of an impulse applied vertically to the surface. Under these conditions the disturbance spreads over the surface in the form of a symmetrical annular wave-system. The initial form of this system will depend on the history of the primitive impulse, but if this be of limited duration, the system gradually develops a characteristic form, marked by three salient features travelling with the velocities proper to irrotational, equivoluminal, and Rayleigh waves, respectively. As the wave-system, thus established, passes any point of the surface, the *horizontal* displacement shows first of all a single well-marked oscillation followed by a period of comparative quiescence, and then another oscillation corresponding to the epoch of arrival of equivoluminal waves. The whole of this stage constitutes what we have called the "minor tremor"; it is, of course, more and more protracted the greater the distance from the source, and its amplitude continually diminishes, not only absolutely but also relatively to that of the "main shock," which we identify with the arrival of the Rayleigh wave. It may be remarked that the history of the minor tremor depends chiefly on the *time-integral* of the primitive impulse; the main shock, on the other hand, follows the time-scale of the primitive impulse, and is affected by every feature of the latter.*

Similar statements apply to the *vertical* displacement, except that the minor tremor leads up more gradually to the main shock, being interrupted, however, by a sort of jerk at the epoch of arrival of equivoluminal waves.

The history of the horizontal and vertical displacements, about the epoch of the main shock, in the case of a typical impulse of the type (165), is shown in fig. 8;

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* Observational evidence in favour of the existence of the three critical epochs in an earthquake disturbance has been collected and discussed by R. D. OLDHAM, "On the Propagation of Earthquake Motion to Great Distances," 'Phil. Trans.,' A, 1900, vol. 194, p. 135.

whilst fig. 9 shows the corresponding orbit of a surface-particle. In fig. 10 a sketch is attempted of the whole progress of the disturbance.

These results are of a fairly definite character, but they are based, as has been said, on purely ideal assumptions, and it remains to inquire how far they are likely to be modified by the actual conditions of the earth. The substitution of an *internal* source for a surface impulse will clearly not affect the general character of the results at a distance great compared with the depth of the source, although differences of detail in the wave-profile at the critical epochs will occur, and we can no longer assume that the disturbance is the same in all vertical planes through the source. Again, the chief qualitative difference introduced by the *curvature* of the earth will be that the minor tremor, whose main features are evidently associated with the outcrop of spherical elastic waves at the surface, will be propagated directly through the earth, so that the first two epochs will (at distances comparable with the radius) be accelerated relatively to the main shock,[*] which being due to the Rayleigh waves will travel, with the velocity proper to these, over the *surface*.[†]

It is a more difficult matter to estimate the nature and extent of the modifications produced by heterogeneity. It is, perhaps, possible to exaggerate these, for the qualitative effect of a *gradual* charge of elastic properties would not be serious, and even considerable discontinuities would have little influence if their scale were small compared with the wave-length[‡] of the primitive impulse. A covering of loose material over the solid rock probably causes only local, though highly irregular, modifications, with some dissipation of energy.

It must be acknowledged that our theoretical curves differ widely in two respects from the records of seismographs. In the first place, they show nothing corresponding to the long successions of to-and-fro vibrations which are characteristic of the latter. It would appear that such indications, so far as they are real and not instrumental, are to be ascribed to a *succession* of primitive shocks, in itself probable enough. Again, the theory gives vertical and horizontal movements of the same order of magnitude, and in the case of the Rayleigh waves, at all events, where a definite comparison can be made, the vertical amplitude is distinctly the greater. The observations, on the other hand, make out the vertical motion to be relatively small. The difficulty must occur on almost any conceivable theory, and appears indeed to be clearly recognised by seismologists, who are accordingly themselves disposed to question the competence of their instruments in this respect.

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[*] *Cf.* R. D. OLDHAM, *loc. cit.*

[†] The theory of free Rayleigh waves on a spherical surface is known; see Professor BROMWICH, *loc. cit.*

[‡] This term is used in the same general sense in which in hydrodynamics we speak of the "length" of a solitary wave travelling along a canal. There is no question, in the present connection, of anything analogous to "oscillatory waves."

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Mallet, J. W.—On the Structure of Gold-Leaf, and the Absorption Spectrum of Gold. Phil. Trans., A, vol. 203, 1904, pp. 43–51.

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II. *On the Structure of Gold-Leaf, and the Absorption Spectrum of Gold.*

By J. W. MALLET, *F.R.S., Professor of Chemistry in the University of Virginia.*

Received May 22,—Read June 11, 1903.

[PLATE 1.]

GOLD-leaf, as seen under the microscope by transmitted light, presents a remarkable appearance which seems to have been hitherto either not at all or only slightly noticed. The colour of the transmitted light is bluish-green, unless silver in considerable proportion be alloyed with the gold; in this latter case the colour is purplish-blue. The amount of light transmitted is, as might be expected, not uniform, the thickness of the gold film varying within very small areas of the surface. All this is well known to anyone who has ever looked through a bit of the leaf.

But, in addition, numerous black lines are visible under very moderate amplification, ramifying irregularly over the surface, here and there showing some tendency to parallelism, but for the most part running into each other in the most irregular way. Fig. 1* illustrates this; it is a microscopic photograph of ordinary commercial gold-leaf, taken with an amplification of 75 diameters, and a distance from the eye-piece to the camera plate of 378 millims.

In FARADAY's Bakerian lecture, read before the Royal Society on February 5, 1857, on the "Experimental Relations of Gold (and other metals) to Light,"† there occur two or three sentences which prove that this peculiar appearance did not escape his keen observation. For example, he says "when the thicker parts of the leaf were examined they seemed to be accumulated plications of the gold, the leaf appearing as a most irregular and crumpled object, with dark veins running across both the thicker and thinner parts, and from one to the other." Again, referring to specimens of gold-leaf which had been heated in oil, he says "it will be seen that it is the thicker folds and parts of the mottled mass that retain the original state longest." And again he remarks, "there is a little difficulty in admitting that such an irregular corrugated film as gold-leaf appears to be, can possess any general compression in one direction only." But FARADAY does not seem to have specially investigated the peculiarity in question, or its cause, and, in view of the process by which gold is extended into these thin films, the terms "plications" and "folds" which he uses must be understood as referring to the appearance only of the leaf and not to its actual structure.

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* All of the microscopic photographs referred to in this paper have been presented to the Royal Society, but only Nos. 4, 6, and 8 have been reproduced for publication.

† 'Phil. Trans.,' 1857, pp. 145–181.

The idea first suggested by the ramification and reticulation of black lines was that they might depend in some way on the crystalline structure of the alloyed gold used for making commercial gold-leaf, modified and distorted during the process of beating. Hence specimens of gold-leaf variously alloyed were compared with each other. The following samples were furnished me by the manufacturers, the W. H. Kemp Company, of 165, Spring Street, New York, with a statement of their composition :—

A. Dark or red gold-leaf, made with an addition of 18 grains of copper to each Troy ounce (480 grains) of pure gold, or, more strictly, gold assaying about 998–999 fine.

B. Gold-leaf of medium colour, made with an addition of 12 grains of copper and 12 grains of silver to each Troy ounce of fine gold.

C. Pale or light-coloured gold-leaf, made with an addition of 6 pennyweights (144 grains) of silver to the Troy ounce of fine gold.

Figs. 1, 2, and 3 show the appearance of these three samples respectively under the amplification already mentioned for No. 1, which represents the gold alloyed with copper only, No. 2 that containing both copper and silver, and No. 3 that containing silver only. The three exhibit some differences, but not much greater than are presented by different samples of leaf of the same composition, and the general character is evidently the same. In consequence of the small amount of light transmitted by the leaf, exposures of the photographic plates for two or three minutes were necessary, and changes in the state of the sky and character of the light during this time prevent the photographs giving quite a correct idea of the different degrees of translucency of the specimens. Owing probably to slight shaking of the floor affecting the position of the camera, the ramified lines do not appear quite as sharp and well defined as when viewed directly through the eye-piece of the microscope.

It was desirable to see whether the same appearance, if referable in any way to the original molecular structure of the metal, would present itself in leaf beaten from pure gold free from all alloy. On applying to two firms of gold-beaters—one in New York and the other in Philadelphia—to make for me a small quantity of leaf from fine gold, I was assured by both that it was impossible to beat pure gold thin enough to be seen through. Dentists' gold foil could be had, but it is quite opaque. The reasons assigned for the difficulty were the excessive tendency of the pure gold to cohere, so that it could not be manipulated without different parts touching each other and sticking together, and also the tendency of the pure metal to stick to the "gold-beaters' skin" or animal membrane used to separate the leaves in beating.

After a good deal of persuasion I succeeded in inducing the manager of the W. H. Kemp Company—Mr. W. R. HANNA—to try the experiment of beating into leaf, as thin as could be had, a sample of fine gold which I sent him. This was "proof gold" from the assay department of the United States Mint at Philadelphia, and therefore of the highest attainable purity. The result was quite satisfactory for the intended purpose, though it would not have been so in a commercial sense, there being a good deal of waste, and many torn leaves and large holes. The microscopic appearance of this pure gold-leaf is shown in fig. 4 (Plate 1). It is in general like the commercial specimens, but the lines are bolder and more strongly defined—a consequence, as I think will be shown, of the greater softness of the pure metal.

Study of these microscopic appearances, and comparison of them with each other and with the micro-photographs of OSMOND, ROBERTS-AUSTEN, ARNOLD, ANDREWS and others, did not seem to support the idea that the lines in question are due to more or less distorted crystalline structure. In order to learn whether the lines are to be referred to, and originate in, the process of gold-beating by which the leaves have been produced, the attempt was made to obtain galvanically-deposited films of something like the same thickness, so that these latter might be microscopically examined by transmitted light.

Pieces of thin rolled silver foil, much larger than would be needed for microscopical examination only, were varnished on one side and then electrolytically coated with fine gold on the other, using a specially prepared pure cyanide solution and an anode of fine gold. As there was no guide by which to determine in advance the thickness of the gold film which would admit of being satisfactorily seen through, the current was passed for various periods of time, producing films of several different thicknesses, and, after the subsequent treatment, one or two were selected which gave the best results. About a square centimetre cut from each piece of foil was well washed with ether to free it from varnish, and was then cemented—the gilded face downwards—upon a slide of thin microscope cover glass by means of Canada balsam somewhat diluted with ether. After time had been afforded for the balsam to harden, the silver was dissolved off slowly by very dilute nitric acid, and the gold film was ready for microscopic examination. Fig. 5 shows the appearance presented, the amplification and distance from eye-piece to camera plate being the same as for fig. 1 and for all the other microscopic illustrations of this paper. It is evident that the mottled structure of this film, showing varying thickness, is unaccompanied by the ramifications of well defined black lines to be seen in beaten gold-leaf. No attention should be given to the two large bars of shadow crossing each other at right angles in this photograph; they are due to the shadow of a part of the window sash having been inadvertently allowed to fall on the illuminating mirror of the microscope.

To test whether the black lines are really due to minute threads or wires of gold with diameters considerably greater than the thickness of those parts of the leaf which can be seen through, it was proposed to protect a piece of gold-leaf by placing it between two sheets of silver foil, roll the whole down to a fraction of the original thickness, remove the silver by means of nitric acid, and see whether the lines in the gold had been broadened out by flattening of the wire-like threads if present. A rectangular piece of fine silver foil, ·019 millim. thick, was folded in two across the middle of its length, a piece of the "fine" gold-leaf which had been specially beaten for me by the W. H. Kemp Company was spread out flat between the two folds of silver, and then by the same firm the whole rolled down until the double thickness, ·038 millim., had been reduced to ·006 millim. Care was taken to introduce the folded edge first between the rolls, so as to prevent as far as possible slipping of one surface of foil upon the other. Examination with nitric acid of different parts of the rolled-down foil showed that, although there had been no small tearing of the gold and many holes had been produced in it, there were quite sufficient areas of it left in a practically continuous state. Assuming that the gold had been rolled out *pari passu* with the silver, each had been reduced to something like one-sixth or one-seventh of the original thickness.

A small piece of the foil in this condition was varnished on one side, and the other side stripped of silver by very dilute nitric acid. A number of specimens were spoiled at this stage, since the acid getting through any holes would attack the silver on the other side and eat its way between the varnish and the gold film, which was so exceedingly thin as not to bear any manipulation when unsupported. A few good specimens, however, were secured. These were cleared of varnish by soaking in ether, cemented by the gold face with diluted Canada balsam to slips of thin microscope cover glass, and, after hardening of the balsam, the second film of silver was gradually removed by very dilute nitric acid. Fig. 6 (Plate 1), representing, under the same amplification as in the other figures, the microscopic appearance of one of these specimens of rolled-down pure gold-leaf, exhibits very distinctly the flattening out of the minute metallic threads, favoured by the greater softness of the gold than of the silver which enclosed it.

As a further test of the black lines being due to minute wires or threads of gold, specimens of the fine gold-leaf were thinned down by partial solution, in order to see whether the lines would remain visible longer than the general surface of the leaf, and the thicker lines longer than the more delicate. The solvent used was a $\frac{1}{2}$ per cent. aqueous solution of potassium cyanide, to which had been added a little hydrogen dioxide. The result is shown in fig. 7, and in fig. 8 (Plate 1), the former of these representing a less, and the latter a more, advanced stage of the solvent attack upon the leaf. The more gradual obliteration of the black lines than of the rest of the surface is quite apparent.

As it seemed to be established that the black lines under examination represent microscopic threads or wires, and that these are developed in the gold during the process of beating, it was natural to look for their possible origin in some corresponding peculiarity of structure in the "gold-beaters' skin" or animal membrane between sheets of which the leaves of gold are extended. But this idea is not borne out by microscopic study of that material. The thin gold foil with which the process is begun is first beaten for about twenty minutes only between surfaces of "cutch" paper, which has simply the structure of a felted mass of vegetable fibres. The principal extension of the gold is brought about by beating for about four hours in a "shoder" or packet of leaves of old or previously often used gold-beaters' skin, the packet, containing a thousand leaves, being from time to time bent between the fingers to loosen the gold films and prevent their sticking to the membrane, and finally by beating for another four hours in a "mould" or similarly made up packet of leaves of new or much less used gold-beaters' skin, repeating the bending of the packet to maintain the looseness of the gold films. The cutch is beaten with hammers of about sixteen pounds in weight, striking about sixty blows a minute, the shoder with hammers of about ten pounds and at the rate of about seventy-five blows per minute, and the mould with six-pound hammers and at the rate of about ninety blows per minute. Figs. 9, 10, and 11 represent respectively the cutch paper, the already much used gold-beaters' skin of the shoder, and the new, or nearly new, skin of the mould. There is nothing in any of these to account for the black lines seen in the gold-leaf. As far as any distant resemblance to these is suggested by some of the vegetable fibres in fig. 9, it is to be remembered that fibres in relief would produce in the gold corresponding furrows, appearing as lines of greater, not less, translucency than that of the rest of the surface. The animal membrane or gold-beaters' skin in which by far the greater part of the beating is done, including all the later part of the work, exhibits in figs. 10 and 11 the simple and nearly uniform structure of the serous coat of the intestine—said to be the cæcum—of the ox which is used for the purpose.

A careful personal inspection of the process of gold-beating at the establishment of the W. H. Kemp Company in New York, has led me to the belief that the production of the ramified lines of microscopic wires or threads in the gold-leaf is due to the following cause. The face of the hammer used is slightly convex, and hence a blow struck with it tends to stretch each sheet of gold, and the animal membrane enclosing it, outwards in all directions from the centre of impact. The membrane is elastic and not absolutely uniform in thickness or tensile strength. Hence it tends to form, along lines of weakness, wrinkles running irregularly outwards, such as may be produced in any stretched piece of cloth by a push of the finger in any given direction. These wrinkles constitute microscopic troughs or furrows into which the soft gold is driven, forming corresponding rods or wires of extremely minute size. The elasticity of the membrane leads to the momentarily developed wrinkles being almost instantly obliterated, while the plasticity of the gold admits of no corresponding disappearance of the wire-like threads produced. The complicated ramification of the lines is no doubt due in part to the irregular distribution of lines of weakness, and therefore of easy stretching, in the membrane, partly to the blows of the hammer falling in rapid succession upon different adjacent parts of the surface, and partly to the lack of uniformity of support given by the other leaves above and below in the packet. The view now stated receives confirmation from a point strongly insisted upon by Mr. HANNA—the very intelligent superintendent of the W. H. Kemp Company's workshops—namely, that for success in the gold-beating process much depends on the condition of the animal membrane as to moisture or dryness. If it be very dry the gold-leaf cracks or breaks, while if the membrane be too moist the leaf sticks to it. The membrane requires to be dried or dampened to correct the opposite effects of change in the atmosphere. This accords with the idea that a certain amount of elastic stretching of the membrane, from which this recovers, is necessary for the permanent or inelastic extension of the gold. In fact, as the area of the gold-leaf is permanently extended by the beating, while that of the membrane is not, the one film manifestly must slide over the other. It is scarcely conceivable that this sliding shall occur at the moment at which a blow falls, when friction between the surfaces is at a maximum. If not, it must occur just afterwards, as a result of the elastic resilience of the membrane, which leaves behind it the plastic gold.

It is evident that the statements to be found in the books as to the actual thickness of gold-leaf—based as they are upon weighing of measured areas—represent only *average* thickness, and that, in view of the decidedly greater thickness of these microscopic threads of gold running through the mass than of the intervening parts, the thickness of these latter parts must be notably less than the average. The following determinations were carefully made with several square decimetres of leaf in each case, accurately measured as to area, and weighed on a delicate assay balance. The results are stated in "microns" (thousandths of a millimetre).

		Average thickness.
Commercial gold-leaf, alloyed with copper only, represented by Fig.	1	·0797μ
,, ,, ,, copper and silver ,,	,, 2	·0822μ
,, ,, ,, silver only ,,	,, 3	·0937μ
Gold-leaf specially beaten from "fine" gold ,,	,, 4	·1082μ
A galvanically deposited film of "fine" gold ,,	,, 5	·1263μ
Maximum thickness of "fine" gold film which can be seen through—about		·2000μ
Dentists' "fine" gold foil		·9228μ

In connection with the microscopic examination of gold films by transmitted light, it seemed to be interesting to make some observations on the absorption spectrum of the metal, especially as there have been recently published the results of spectroscopic study of the light which the metal reflects.

It was proposed to examine for this purpose metallic gold in the following forms :—

1. Pure or "fine" gold-leaf.

2. Gold chemically reduced in a dilute aqueous solution of its chloride—so-called colloidal gold—the metal being in sufficient quantity and state of aggregation to transmit greenish-blue light.
3. Ditto, a less amount of more finely distributed gold transmitting ruby-red light.
4. Glass coloured by gold so as to transmit greenish-blue light—the so-called saphirine glass.
5. Glass coloured ruby-red by very finely divided gold.

It was found to be impracticable to secure any result for the gold-leaf, on account of the very small amount of light transmitted. For the colloidal gold in water, and the gold-coloured glass, the following results were obtained.

Visible Spectrum.

The source of light was a strong electric arc between closely placed carbon poles, with a slit of about $\frac{1}{2}$ millim. in width. Dispersion was obtained by a Rowland concave grating of 21·5 feet focal length, ruled with about 15,000 lines to the inch, using the spectrum of the first order. The photographs were taken on M. A. Seed dry plates ("orthochromatic, L"), specially sensitized for the region from green to red inclusive. The original photographs were laid side by side, so that the positions of like wave-lengths were the same for all, and then re-photographed together on a reduced scale. The results are shown in fig. 12, with a few positions indicated in Ångström units.

Taking the strips in order from the top downwards, the first (uppermost) strip represents the light transmitted simply through a sheet, about 2 millims. thick, of colourless glass of the same kind as that on which the gold ruby-red is "flashed," and which also formed the end plates of cells containing the colloidal gold in aqueous suspension—time of exposure about 2 minutes—the darkness at the less refrangible end is due, not to absorption, but to the insensitiveness of the photographic film for rays in this region. The second strip shows the effect of transmission through a column of water, 2·25 centims. long, containing 75 milligs. of metallic gold to the litre, reduced from the chloride by potassium acid carbonate and formic aldehyde, and exhibiting dark greenish-blue colour—time of exposure 30 minutes. The third represents also colloidal gold in watery suspension, but in a column of 9·25 centims. long, with 50 milligs. of gold per litre, and showing a blue or slightly violet-blue colour—time of exposure 20 minutes. The fourth represents the same, in a column of same length as the last, but with only 20 milligs. of gold per litre, and showing a clear ruby-red colour—time of exposure 10 minutes. The fifth, shifted over to the right to secure correspondence of position for equal wave-lengths, is the same as the first, but with shorter exposure; the right-hand end is in the region of slight sensitiveness of the film. The sixth (lowermost) strip shows the result of transmission through the "flashed" ruby-red glass, with very long exposure—1 hour and 10 minutes.

In these photographs there is no indication of well defined absorption bands. The general absorption belongs mainly to the middle portion of the spectrum, and is, on the whole, more marked at the less refrangible end, with notable increase of absorption in this region as the amount of gold present is increased. The position of maximum absorption is nearer to the long-wave end for the glass than for the colloidal gold in water. It is interesting to note that, while no photographed results could be obtained from the saphirine glass, the absorption being too far in the red for the sensitiveness of the film, eye observation of this glass, using sunlight and a glass-prism spectroscope, showed a distinct belt of absorption extending from about 5700 to 6250, beside the general absorption of rays of shorter wave-length. Allowance has to be made in the photographs for insensitiveness of the film at the red end of the spectrum.

Ultra-violet Spectrum.

This was examined with a quartz prism, and for the liquids a tube closed at the ends by plates of quartz. The source of light was electric sparks between cadmium poles placed pretty near each other. The results are shown in figs. 13, 14 and 15, a few of the positions being indicated by the wave-lengths of the cadmium lines, as before. No results could be obtained for the saphirine or the ruby glass, the glass alone absorbing all rays in the ultra violet. Fig. 13 represents the water with colloidal gold in suspension, 75 milligs. to the litre, in a column of 2·25 centims. long. Fig. 14 represents a like liquid, with 50 milligs. per litre, and in a column 9·25 centims. long. Fig. 15 is the same, with 20 milligs. per litre, and in a column also 9·25 centims. long.

In each of these three figures the three uppermost strips represent exposures for 3, 5 and 10 *minutes* respectively (counting from above downwards), the light passing through the colloidal gold liquid, while the four lower strips exhibit the results from sparks through air (no gold liquid interposed) for 1, 2, 5 and 10 *seconds* respectively. The general absorption, without indication of dark bands, begins to be well marked at about 3500, and increases toward the more refrangible portion of the spectrum, the effect increasing also with the amount of gold present.

Infra-red Spectrum.

This was examined, by the obliging permission of Professor S. P. LANGLEY, Secretary of the Smithsonian Institution, Washington, D.C., in the astrophysical laboratory of that institution, using sunlight, a rock-salt prism, and Professor

Langley's bolometer with photographic auto-registration of the results. These results were in general as follows :—

The specimen of ruby-red gold glass almost totally absorbs the light of the short-wave-length side of D, rapidly increases to the full transparency of the ordinary colourless (unflashed) glass at about A, and continues as transparent as this ordinary glass to about $2{\cdot}5\mu$. The saphirine or blue glass coloured by gold, cuts off the light to near C, then rises very rapidly to great transparency at and beyond A.

The red colloidal gold liquid No. 1, 20 milligs. metallic gold per litre, contained in a cell with end plates of thin microscope glass, 4·5 centims. apart, produces great general absorption in the visible spectrum, though not reaching to the point of completely, or almost completely, extinguishing any rays included within the region of spectrum studied, as was the case with the glass specimens. The absorption of this liquid becomes practically identical with that of distilled water at and beyond A.

The violet-blue colloidal gold liquid No. 2, 50 milligs. gold per litre, and the greenish-blue liquid No. 3, 75 milligs. gold per litre, behave on the whole like liquid No. 1, except that they diminish the radiations throughout the spectrum to a very great extent, as if by the interposition of opaque obstacles to the rays. Liquid No. 3 appears relatively less transparent in the visible spectrum, besides being generally less transparent throughout the spectrum.

I regret that the blue-print tracings of the bolometer curves are so faint as not to allow of photographic reproduction on a reduced scale.

For the microscopic and spectroscopic photographs I have to thank the kind assistance of Professor A. H. Tuttle and Dr. W. J. Humphreys of this University.

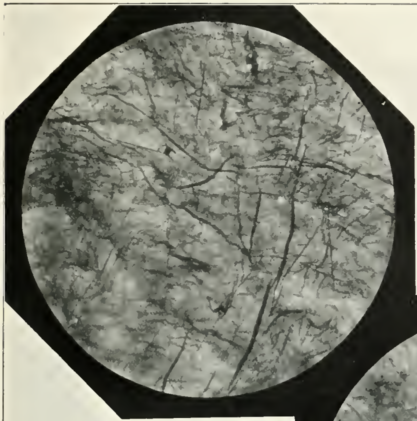
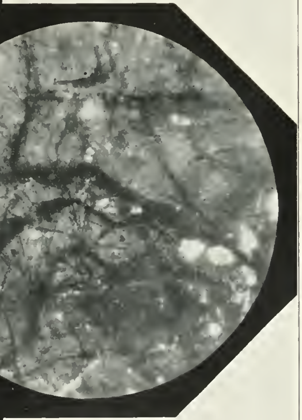


Fig. 4.

Fig. 6.



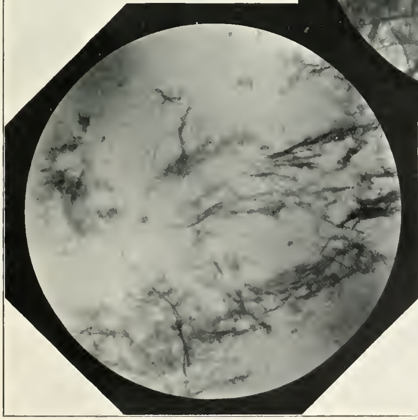


Fig. 8.

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III. *Mathematical Contributions to the Theory of Evolution.*—XII. *On a Generalised Theory of Alternative Inheritance, with special Reference to* MENDEL'S *Laws.*

By KARL PEARSON, *F.R.S.*

Received September 11,—Read November 26, 1903.

(1.) *Introductory. On a Generalised Theory of Alternative Inheritance, with special Reference to* MENDEL'S *Laws.*[*]

IT seems likely to be of interest at the present time to consider rather at length a fairly full mathematical theory of the pure gamete. We do not venture to call this theory a generalised Mendelian theory of inheritance, partly because it is not even the most general theory of the pure gamete conceivable, partly because MENDEL'S original theory of heredity was perfectly clear and perfectly simple, and is not the theory here developed. The pure and simple Mendelian theory seems to have been discarded in the light of recent experimental results by more than one Mendelian, both in this country and abroad. The original Mendelian theory has been replaced by what are termed "Mendelian Principles." In this aspect of investigation the fundamental principles propounded by MENDEL are given up, and for each individual case a pure gamete formula of one kind or another is suggested as describing the facts.[†] This formula is then emphasised, modified or discarded, according as it fits well, badly, or not at all with the growing mass of experimental data.

It is quite clear that it is impossible while this process is going on to term anything whatever Mendelian as far as theory is concerned. The present investigation is therefore not a generalised Mendelian theory of heredity: we speak of it merely as a generalised theory of alternative inheritance, and it is based on the conception that the gamete remains pure, and that the gametes of two groups, while they may link up to form a complete zygote, do not thereby absolutely fuse and lose their

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[*] I owe the incentive to this memoir to Professor W. F. R. WELDON, who had already worked at some of the simpler special cases and who placed his results entirely at my disposal.

[†] See especially TSCHERMAK, 'Zeitschrift f. d. landwirthsch. Versuchswesen,' Jahrg. IV. ("Ueber Züchtung neuer Getreiderassen"); DE VRIES, 'Ber. d. deutsch. botan. Gesellsch.,' vol. xviii. (1900), pp. 435–443; BATESON, 'Proc. Camb. Phil. Soc.,' vol. 12, p. 53; 'Nature,' April and May, 1903.

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identity. The analytical expression of this is represented by the fundamental formula:

$$\left.\begin{array}{r}(\mathrm{AA}') \times (aa') \\ = (\mathrm{A} + \mathrm{A}')(a + a)\end{array}\right\} = \left\{\begin{array}{l}\mathrm{A}a \\ \mathrm{A}a' \\ \mathrm{A}'a \\ \mathrm{A}'a'\end{array}\right. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)},$$

where (AA') and (aa') are the parental zygotes, and the right-hand side of the equation represents the four possible constitutions of the offspring. Such a formula as the above may be accepted without any hypothesis as to dominant and recessive characters, but these terms were certainly essential to Mendelian theory as propounded by MENDEL himself, and it becomes very doubtful whether we ought to attach his name to any theory which discards these "recognition marks." It is very convenient, however, to have names for the alternative elements expressed by capital and small letters respectively. I propose for the purpose of this paper to term an A-element a *protogene*, and an a-element an *allogene*. Two protogenic elements will give rise to a protogenic zygote AA, two allogenic elements to an allogenic zygote aa, and a protogenic and allogenic element to what Mr. BATESON has termed a heterozygote Aa. We may thus class his homozygotes into protozygotes and allozygotes. We reach pure Mendelianism by making our protozygotes "dominants," our allozygotes, "recessives," and our heterozygotes "hybrids of dominant character." In so far as our theory of pure gametes replaces protozygote, allozygote, and heterozygote by "dominant," "recessive," and "hybrid with dominant character," it becomes a generalised Mendelian theory, but only in this case. Otherwise we must look upon it as an attempt—in one direction only of course—to give a consistent mathematical basis to the various formulæ which have been propounded for describing statistical data classed under Mendelian categories; shortly we shall endeavour to develop a general pure gamete theory.

The results were worked out in a purely impartial frame of mind; indeed, once state the hypotheses, and the analysis is far too complex to allow us to predict *à priori* what can possibly result from it, nor does the investigation admit of any but one solution. If the hypotheses are admissible, then any narrower pure gamete formula must lead to results embraced under our general conclusions.

What we have to admit at the present time are the following conditions:—

(i.) The existence of a vast bulk of evidence that heredity, as far as measurable characters are concerned, follows within a population perfectly definite laws.

(ii.) The existence of another mass of experiments, in which simple and pure Mendelianism is certainly inadmissible, but in which certain ratios undoubtedly approach the values they would have on such a simple and pure Mendelian theory.

It is possible, therefore, that a generalised theory of the pure gamete would account for (ii.); it can only do so satisfactorily, however, if it does not contradict the results of (i.). Hence arises the present attempt to develop in one direction a generalised theory of the action of the germ-cell.

As we have frequently had to assert, the laws investigated under (i.) have nothing whatever to do with any physiological hypothesis. That a physiological hypothesis leads to them is not much test of its validity—it is a necessary, but not *sufficient*, criterion of its correctness. If, however, it contradicts them, we are bound to discard it, and seek for its modification or replacement. The present study is an attempt to see how far one generalised pure gamete theory leads to results in accordance with the law of regression and the known nature of the distributions of offspring in populations.

(2.) *Nature of Hypothesis adopted.*

We start with a zygote consisting not of a single protogenic pair AA, but built up of n such pairs,

$$A_1A_1 + A_2A_2 + A_3A_3 + \ldots + A_nA_n.$$

We suppose this to produce gametes which unite with those of a similar allogenic zygote

$$a_1a_1 + a_2a_2 + a_3a_3 + \ldots + a_na_n.$$

Any element of the protogenic gamete must unite with the corresponding element of the allogenic gamete, *i.e.*, A_r with a_r, and by the fundamental principle (i.) above, this gives rise to the four possibilities

$$A_ra_r\,,$$
$$A_ra_r\,,$$
$$a_rA_r\,,$$
$$a_rA_r\,,$$

which are all of the same constitution. The result is the hybrid group, symbolised by

$$a_1A_1 + a_2A_2 + a_3A_3 + \ldots + a_nA_n,$$

the perfect multiple heterozygote.

The population will now be supposed to consist of any number of such perfect heterozygotes, which we shall suppose to again cross. We shall now have

$$a_rA_r \times a_rA_r = \begin{cases} a_ra_r \\ a_rA_r \\ A_ra_r \\ A_rA_r\,, \end{cases}$$

or, each couplet will give rise to four possibilities, representing, however, only three constitutional differences, expressed by

$$a_r a_r + 2a_r \mathrm{A}_r + \mathrm{A}_r \mathrm{A}_r.$$

Since these four possibilities may occur with each of the n couplets, we shall have when two perfect heterozygotes cross, 4^n resulting possibilities. These form the resulting population of the second generation. Our first problem will be to find the distribution of this population. This, according to MENDEL, is the segregating generation. We must inquire into the frequency of each constitutional difference in this segregating generation.

We now reach our second limiting hypothesis, which is needful if we are to apply our theory to sexual reproduction. We suppose :—

There to be an absence of homogamy (including self-fertilisation), and the members of the second generation to cross absolutely at random and with equal fertility.* We have then to ask what is the distribution of constitutional differences in the third generation. Does the process of segregation begun in the second generation continue to the third, or does the population now remain stable? Is the *continual* segregation into pure protogenic and allogenic individuals a *necessary* result of any pure gamete theory, or does the belief in such necessity depend upon the first Mendelian experimenters working only with self-fertilising individuals?

(3.) PROBLEM I.—*To find the Distribution of the Offspring of the Perfect Heterozygotes.*

We shall here use a symbolic form of analysis. Let u stand for aa, v for $a\mathrm{A}$, and w for AA; then any corresponding couplets will give rise to

$$u + 2v + w,$$

and any one of these constitutions may be associated with one of the similar constitutions in any of the remaining $n - 1$ couplets. Hence the general distribution of the population will be given by the terms of the multinomial

$$(u + 2v + w)^n.$$

This equals†

$$u^n + nu^{n-1}(2v + w) + \frac{n(n-1)}{1 \cdot 2} u^{n-2}(2v + w)^2 + \ldots + c_{n,s,0}\, u^{n-s}(2v + w)^s + \ldots$$

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* If one is to study heredity in populations with a view to the problem of evolution, the conditions as to fertilisation should approach as far as possible the conditions we suppose them to be under in a natural state; we must fix our attention on the *mass* relations between successive generations of the population.

† Throughout this memoir the symbol $c_{n,p,q}$ is used for the expression $\underline{|n} / \{\underline{|n-p-q}\ \underline{|p}\ \underline{|q}\}$.

Thus, for example, there would be out of the total population of possibilities 4^n: 1 purely allogenic individual, $n \times 3$ individuals with $n - 1$ allogenic couplets; $2n$ of these would have one heterogenic couplet, and n would have one protogenic couplet.

Generally there would be $3^s\ c_{n,s,0}$ individuals with $n - s$ allogenic couplets, and these individuals would be distributed according to the terms of the binomial $(2v + w)^s$.

We are thus able to write down at once the number of any class of individual that can appear in the segregating generation. For example, how often do individuals like $u^{n-p-q}v^pw^q$ appear, *i.e.*, individuals with $n - p - q$ allogenic, p heterogenic, and q protogenic couplets?

To answer this problem all we have to do is to pick out the coefficient of $u^{n-p-q}v^pw^q$ in the above multinomial, and the result is

$$2^p c_{n,\,p,\,q}.$$

We are thus fully able to predict how many individuals of each kind ought to occur when a population of perfect n-couplet heterozygotes are crossed.

Corollary (i.).—Let us consider only the number of allogenic couplets in the distribution of the segregating generation. If we were "pure Mendelians" we should for the purpose of character classification make $v = w$, as the heterogenic couplet would then give the dominant character. But without doing this we can assume v and w to be non-u's.

Hence the distribution of allogenic characters in the population follows the simple binomial

$$4^n\left(\tfrac{1}{4} + \tfrac{3}{4}\right)^n.$$

Thus we see that the distribution would be a skew binomial closely approximating to my skew curve of Type III.,[*] and becoming indefinitely close to a normal distribution of the form

$$y = y_0 e^{-x^2/2\sigma^2},$$

when the number of couplets, n, is indefinitely increased.

For any value of n the mean of the skew binomial as measured by the formula of the memoir on "Skew Variation,"[†]

$$= n + 1 - \left(1 + \tfrac{3}{4}n\right) = \tfrac{1}{4}n,$$

and the standard deviation $= \sqrt{\tfrac{3}{16}n}$.

Thus the mean number of allogenic couplets in the members of the segregating generation is $\tfrac{1}{4}$ of the total number of couplets.

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[*] 'Phil. Trans.,' A, vol. 186, p. 373.

[†] *Ibid.*, p. 346.

Corollary (ii.).—The distribution of heterogenic couplets in the segregating generation is given by the symmetrical binomial

$$4^n\left(\tfrac{2}{4}+\tfrac{2}{4}\right)^n.$$

The mean number is therefore $\frac{1}{2}n$, and the standard deviation $= \sqrt{\frac{1}{4}n}$.

This is a symmetrical binomial, and approaches extremely closely, even for a fairly small value of n, to the normal curve. We see that if any character depends solely upon heterogenic couplets, the distribution will be nearly normal, and the variability slightly greater than one depending on the allogenic (or, of course, the protogenic) couplets only.

To sum up, then, so far as the distribution of characters depending upon allogenic or heterogenic couplets goes, we may say that a generalised theory of the pure gamete leads us to those normal and skew distributions of frequency with which biometric studies of variability have made us already familiar. It would not be possible to base a crucial experiment on the existence or non-existence of such frequency distributions. The generalised pure gamete theory would, however, account for their appearance, which, of course, a purely descriptive statistical theory cannot do. On the other hand, distributions diverging much beyond the errors of random sampling from binomials of the above types would tell *pro tatno* against the pure gamete theory in its above form. The presence of binomials of two types only, $(\frac{1}{2}+\frac{1}{2})^n$ and $(\frac{1}{4}+\frac{3}{4})^n$, ought to be capable of detection, even if it would not already have been discovered, had it been the rule.

(4.) PROPOSITION II.—*To determine the Distribution of the Offspring of the Segregating Generation, supposing them to Mate at Random and without Differential Fertility.*

The solution of this problem may be reached as follows:—

Suppose P any male, and Q any female, say each of $n-1$ couplets, producing an array of offspring, which we will denote by R; now suppose an additional couplet, the n^{th}, added to both male and female zygote. The male may be now:

$$P + a_n a_n,\quad \text{or}\quad P + a_n A_n,\quad \text{or}\quad P + A_n a_n,\quad \text{or}\quad P + A_n A_n\,;$$

and the female may be

$$Q + a_n a_n,\quad \text{or}\quad Q + a_n A_n,\quad \text{or}\quad Q + A_n a_n,\quad \text{or}\quad Q + A_n A_n\,;$$

that is, we get 4×4 new mating individuals, with $4 \times 4 \times 4$ new offspring possibilities.

Now consider the first father $P + a_n a_n$; the possibilities which arise from mating him with the four mothers are the array R of offspring combined with any one of the

16 possibilities $8a_n a_n + 8a_n A_n$, or this is the same thing as multiplying the R array by the symbolic factor $8(u + v) = 16U$, say.

The next pair of fathers $2(P + a_n A_n)$ with the four mothers reproduce the array R of offspring combined with $8(a_n a_n + 2a_n A_n + A_n A_n)$, or 32 possibilities. But this is the same as multiplying the R-array by the symbolic factor $8(u + 2v + w) = 32V$, say. Lastly, the $P + A_n A_n$ father with the four mothers gives 16 possibilities of the form $8a_n A_n + 8A_n A_n$ to be combined with the R-array of offspring, which is the same thing as multiplying the R-array by the symbolic factor $8(v + w) = 16W$, say.

We have at once the symbolic relation among the operators:

$$U + 2V + W = u + 2v + w;$$

and, further, the important result that the array of offspring due to any pair P and Q of $\overline{n-1}$-couplet parents can be converted into the arrays of offspring due to the 16 pairs of parents formed by adding an additional couplet to P and Q, by multiplying that array by the symbolic factor

$$16U + 32V + 16W = 16(u + 2v + w).$$

We have thus by induction a means of finding the array of offspring due to a population of parents of n couplets from the series of arrays due to a population of n − 1 couplets. Since all the arrays are to be multiplied by the same symbolic factor, we can multiply their total by this factor. Or the distribution of offspring of $(n - 1)$-couplet parents being J, that of n-couplet parents

$$= 16(u + 2v + w)J = 4 \times 4 \times 4 \,.\, (\tfrac{1}{4}u + \tfrac{2}{4}v + \tfrac{1}{4}w)J.$$

Now consider parents of one couplet, their distribution is given by $aa + 2aA + AA$, and they are to mate with the same series, $aa + 2aA + AA$.

But

$$aa \times aa = 4aa,$$

$$2(aa \times 2aA) = 2(4aa + 4aA),$$

$$2(aa \times AA) = 2(4aA),$$

$$2aA \times 2aA = 4aa + 8aA + 4AA,$$

$$2(2aA \times AA) = 2(4aA + 4AA),$$

$$AA \times AA = 4AA,$$

$$\text{Total} = 16aa + 32aA + 16AA.$$

$$= 16(u + 2v + w) = 4 \times 4 \times 4(\tfrac{1}{4}u + \tfrac{2}{4}v + \tfrac{1}{4}w)$$

symbolically.

Hence, by the above proposition, the distribution of offspring of parents of two couplets is

$$4 \times 4 \times 4 \,.\, (\tfrac{1}{4}u + \tfrac{2}{4}v + \tfrac{1}{4}w) \times 4 \times 4 \times 4 \,.\, (\tfrac{1}{4}u + \tfrac{2}{4}v + \tfrac{1}{4}w)$$
$$= 4^2 \times 4^2 \times 4^2 \,.\, (\tfrac{1}{4}u + \tfrac{2}{4}v + \tfrac{1}{4}w)^2,$$

and, by induction, the distribution of offspring for the random mating of parents of n couplets is

$$4^n \times 4^n \times 4^n \,.\, (\tfrac{1}{4}u + \tfrac{2}{4}v + \tfrac{1}{4}w)^n.$$

This, except for the constant factor $4^n \times 4^n$, is absolutely identical with the distribution of the parental population, and accordingly if the next generation also mates at random, the mixed race will continue to reproduce itself without change. We therefore reach the following result:—

However many couplets we suppose the character under investigation to depend upon, the offspring of the hybrids—or the segregating generation—if they breed at random inter se, *will not segregate further, but continue to reproduce themselves in the same proportions as a stable population.*

It is thus clear that the apparent want of stability in a Mendelian population, the continued segregation and ultimate disappearance of the heterozygotes, is solely a result of self-fertilisation; with random cross fertilisation there is no disappearance of any class whatever in the offspring of the hybrids, but each class continues to be reproduced in the same proportions. Thus our generalised theory lends no countenance to the appearance of any "mutations" within a hybrid population under random mating; the only appearance of new constitutions is in the segregating generation, or the first generation of hybrid offspring. Except at this stage, the appearance of the unfamiliar is only the chance occurrence of a very rare normal variation. When we recollect that a purely allogenic individual is only to be expected once in a population of 4^n individuals, or if there be ten couplets, once in more than a million individuals, it will be clearly seen that the rarity of some of the more exceptional normal constitutions may easily lead to their being looked upon as "mutations," even if they appear in the offspring of a population many generations removed from hybridisation.

(5.) PROPOSITION III.—*To find the Array of Offspring due to a Parent of given Gametic Constitution mating at Random.*

This can be again deduced by the method of induction adopted in the last proposition.

Supposing a male P of $n - 1$ couplets to mate with all possible females, and R_{n-1} to be the array of offspring, then we have seen in the last proposition that if we add an n^{th} couplet $a_n a_n$ to P, the array of offspring due to $P + a_n a_n$ will be $16UR_{n-1}$; if we add a couplet $a_n A_n$, the array of offspring due to fathers of type $P + a_n A_n$ will be

16VR_{n-1}, and if we add a couplet of form A_nA_n, the array will be of the form 8WR_{n-1}. Now start with a father of one couplet; this must be a_1a_1, or $a_1\text{A}_1$, or A_1A_1. or in our symbolic notation u, v, or w; the offspring array are respectively $8a_1a_1 + 8a_1\text{A}_1$ or $4a_1a_1 + 8a_1\text{A}_1 + 4\text{A}_1\text{A}_1$, or $8a_1\text{A}_1 + 8\text{A}_1\text{A}_1$, *i.e.*, 16U, 16V, or 16W. These, therefore, are the possible values of R_1. Hence, by the principle just developed above, the array of offspring due to a father of type

$$u^{n-p-q}\,v^p\,w^q$$

is

$$(16\text{U})^{n-p-q}\,(16\text{V})^p\,(16\text{W})^q,$$

or remembering that such fathers occur with a frequency of $2^p c_{n,p\,q}$, we have for the total distribution of offspring of all fathers of type

$$u^{n-p-q}\,v^p\,w^q,$$

the symbolic result

$$4^n \times 4^n \,.\, c_{n,p,q}\text{U}^{n-p-q}\,(2\text{V})^p\text{W}^q.$$

Substituting, the following expression would give all offspring of fathers of the type $u^{n-p-q}v^pw^q$, *i.e.*, with $n-p-q$ allogenic, p heterogenic, and q protogenic couplets

$$4^n \times 4^n \,.\, c_{n,p,q}\,(\tfrac{1}{2}u + \tfrac{1}{2}v)^{n-p-q}\,(\tfrac{1}{2}u + v + \tfrac{1}{2}w)^p\,(\tfrac{1}{2}v + \tfrac{1}{2}w)^q.$$

Therefore, given n and given p and q, it is merely a matter of expansion to find the array of offspring due to any special class of father.

Corollary (i.).—So far we have supposed our special class of father to be defined by the exact couplet distribution constitutional to him. But it is of interest to consider the array of offspring we get supposing only the allogenic couplets fixed in number, for example, in a generalised Mendelian theory if the number of recessive couplets be fixed, but the heterogenic and dominant, as both exhibiting dominant characters, be considered as indifferent. Let s = number of allogenic couplets, then we have to sum all arrays like

$$4^n \times 4^n \,.\, c_{n,p,q}\text{U}^s\,(2\text{V})^p\text{W}^q,$$

subject to the condition that $p + q = n - s$.

The result is clearly

$$\begin{aligned} &4^n \times 4^n \,.\, c_{n,s,o}\,\Sigma\,\{c_{p+q,p,o}\,(2\text{V})^p\text{W}^q\} \\ &= 4^n \times 4^n \,.\, c_{n,s,o}\text{U}^s\,(2\text{V} + \text{W})^{n-s} \\ &= 4^n \times 4^n \,.\, c_{n,s,o}\,(\tfrac{1}{2}u + \tfrac{1}{2}v)^s\,(\tfrac{1}{2}u + \tfrac{3}{2}v + w)^{n-s} \\ &= 4^n \times 4^n \,.\, c_{n,s,o}\,(\tfrac{1}{2}u + \tfrac{1}{2}v)^s\,\{(\tfrac{1}{2}u + \tfrac{1}{2}v) + (v + w)\}^{n-s}. \end{aligned}$$

This, we note, is not a pure binomial, or the arrays of offspring of a father with a given allogenic constitution are not either symmetrical or skew binomials, but of a much more complex character. The only exception is the array of offspring of pure allogenic fathers,[*] which is given by

$$4^n \times 4^n \times (\tfrac{1}{2}u + \tfrac{1}{2}v)^n.$$

This is a symmetrical binomial. This result is, of course, of special interest, for it gives us the distribution of offspring if the hybrid offspring were at any time crossed with the pure allogenic race, which was one of the original factors of the hybridisation. The deviation from binomial distribution in the above arrays ought to be further considered, for if this deviation should turn out to be very significant, it would form a convenient test for any generalised theory of pure gametes.

Corollary (ii.).—If we sum the above expressions for the array of offspring of all fathers of p allogenic couplets for values of s from o to n, we have the total offspring population

$$\begin{aligned} &= 4^n \times 4^n \,.\, \Sigma c_{p,\,s,\,o} \mathrm{U}^s (2\mathrm{V} + \mathrm{W})^{n-s} \\ &= 4^n \times 4^n \times (\mathrm{U} + 2\mathrm{V} + \mathrm{W})^n \\ &= 4^n \times 4^n \times (u + 2v + w)^n, \end{aligned}$$

a result we have already found in Proposition II. as giving the distribution of the total offspring population.

(6.) PROPOSITION IV.—*To find the Mean Number of Allogenic Couplets in the Offspring of all Fathers having in their Constitution s-allogenic Couplets.*

By the first corollary to the last proposition the distribution of such offspring is given by

$$4^n \times 4^n \,.\, c_{p,\,s,\,o} (\tfrac{1}{2}u + \tfrac{1}{2}v)^s \{\tfrac{1}{2}u + \tfrac{1}{2}v + 2\eta\}^{n-s},$$

where η is written for $\frac{1}{2}(v + w)$, a quantity which is unity so long as we consider not the distribution, but the total number of the non-allogenic couplets. Now this is clearly the sum of a number of symmetrical binomials in $\frac{1}{2}u + \frac{1}{2}v$, and may be put

$$= 4^n \times 4^n \,.\, \sum_{i=o}^{i=p-s} c_{p,\,s,\,i} (\tfrac{1}{2}u + \tfrac{1}{2}v)^{n-i} (2\eta)^i.$$

Now the means of each of these binomials can be found from the general theory of the binomial.[†] If we take our origin at $\overline{n+1}$ allogenic couplets, with a frequency zero, the mean of the first binomial, or

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[*] Or, of course, the array of sons from pure protogenic fathers.

[†] 'Phil. Trans.,' A, vol. 186, p. 373.

$(\frac{1}{2}u+\frac{1}{2}v)^n$ is at $1+\frac{1}{2}n$, and its total frequency $f_1 = 4^n \times 4^n \times c_{n,s,o}$;

the mean of the second binomial, or

$(\frac{1}{2}u+\frac{1}{2}v)^{n-1}$ . . $2+\frac{1}{2}(n-1)$. . . . . $f_2 = \frac{n-s}{1} 2f_1$;

the mean of the third binomial, or

$(\frac{1}{2}u+\frac{1}{2}v)^{n-2}$ . . $3+\frac{1}{2}(n-2)$. . . . . $f_3 = \frac{(n-s)(n-s-1)}{1\,.\,2} 2^2 f_1$;

the mean of the $(i+1)^{\text{th}}$ binomial

$(\frac{1}{2}u+\frac{1}{2}v)^{n-i}$ . . $i+1+\frac{1}{2}(n-i)$. $f_{i+1} = \frac{(n-s)(n-s-1)\ldots(n-s-i+1)}{1\,.\,2\,.\,3\ldots i} 2^i f_1$.

The total frequency is accordingly

$$F_s = f_1 + f_2 + f_3 + \ldots = f_1(1+2)^{n-3}$$
$$= 4^n \times 4^n c_{n,s,o} 3^{n-s}.$$

Hence if m_s be the distance from the same origin of the mean of the above system of binomials

$$f_1 \times 3^{n-s} \times m_s = f_1\left(1+\frac{n}{2}\right) + f_2\left(2+\frac{n-1}{2}\right) + \ldots + f_{i+1}\left(i+1+\frac{n-i}{2}\right) + \ldots$$
$$= f_1\left\{1+\frac{n}{2}+2(n-s)\left(2+\frac{n-1}{2}\right)+2^2\frac{(n-s)(n-s-1)}{1\,.\,2}\left(3+\frac{n-2}{2}\right)\right.$$
$$\left.+\ldots+2^i\frac{(n-s)(n-s-1)\ldots(n-s-i+1)}{1.2.3\ldots i}\left(i+1+\frac{n-i}{2}\right)+\ldots\right\}.$$

Now

$$(1+2x)^{n-s} = \left\{1+2(n-s)x+2^2\frac{(n-s)(n-s-1)}{1\,.\,2}x^2+\ldots\right\}.$$

Multiply by x^2, differentiate both sides and divide by 2, finally putting $x = 1$, and we find

$$3^{n-s}+(n-s)3^{n-s-1} = 1+2(n-s)\tfrac{3}{2}+2^2\frac{(n-s)(n-s-1)}{1\,.\,2}\tfrac{4}{2}+\ldots$$
$$+2^i\frac{(n-s)(n-s-1)\ldots(n-s-i+1)}{1\,.\,2\,.\,3\ldots i}\,\frac{i+2}{2}+\text{\&c.}$$

Hence we deduce

$$f_1 \times 3^{n-s} m_s = f_1\{3^{n-s}+(n-s)3^{n-s-1}+\tfrac{1}{2}n3^{n-s}\},$$

or

$$m_s = 1+\tfrac{1}{2}n+\tfrac{1}{3}(n-s).$$

But the mean of the whole population of offspring is at $1 + \frac{3}{4}n$ from our origin. Thus we have the final results:

Mean number of allogenic couplets in offspring of fathers with s allogenic couplets

$$= \tfrac{1}{2}n - \tfrac{1}{3}(n - s) \text{ allogenic couplets.}$$

Deviation from mean of general population of this array of offspring

$$= \tfrac{1}{4}n - \tfrac{1}{3}(n - s) = \tfrac{1}{12}(4s - n).$$

Deviation of fathers from mean of population

$$= s - \tfrac{1}{4}n = \tfrac{1}{4}(4s - n).$$

Thus

$$\frac{\text{Deviation of offspring from mean of population}}{\text{Deviation of fathers from mean of population}} = \tfrac{1}{3}.$$

We have then the following results, which could certainly not have been foreseen:—

(*a.*) The regression is constant for all arrays, or the regression curve is a straight line.

(*b.*) The slope of this straight line is $\frac{1}{3}$, or, since we have seen that the population is stable, the parental correlation is $\frac{1}{3}$ also.

Now these results are of very singular importance. A very general theory of the pure gamete type leads to linearity of the regression curve, a result amply verified by observations on inheritance in populations;* and this result is quite independent of the number of couplets supposed to form the total character of the parent, or of the fact that in this case the arrays of offspring are skew and do not obey the normal law.† Further, the value of the correlation reached is numerically identical with the value obtained by FRANCIS GALTON in his original investigations on the inheritance of stature! The generalised theory of the pure gamete is thus shown, whatever the number of couplets taken, to lead to precisely the chief results already obtained by those who have studied heredity statistically. So far then it might appear that a generalised theory of the pure gamete was capable of being brought into accordance with the chief results of biometric experience in heredity. This would undoubtedly be a great step forward, as linking up perfectly definite inheritance results with a physiological theory of heredity. Unfortunately the whole drift of recent biometric observations on heredity emphasises three points:

First.—That the parental correlation appears to be markedly greater than $\frac{1}{3}$, nearer to ·45 to ·5.

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* GALTON, 'Natural Inheritance,' p. 96: 'Biometrika,' vol. 2, pp. 216 and 362–3.

† This is further demonstration that linearity of regression has nothing whatever to do with the Gauss-Laplacian law of errors, *i.e.*, normal curves or surfaces.

Secondly.—That this correlation appears to vary slightly from character to character.

Thirdly.—That it does not appear to be absolutely the same for all species.

It is most unfortunate for this general theory of the pure gamete, that it throws the Mendelian back into the position of the biometrician of 1885.[*] One might have hoped that the generality involved in n couplets would have led to the requisite elasticity, or, failing this, to a numerical value of parental correlation nearer the cluster point of existing measurements than $\frac{1}{3}$. We can only say, at present, that a generalised theory of the pure gamete leads to precisely the same general features of regression as have been observed by the biometricians, but it appears numerically too narrow to describe the observed facts.

(7.) PROPOSITION V.—*To find the Standard Deviation of the Array of Offspring due to Fathers with s-allogenic Couplets.*

We have to find the standard deviation σ_s of the combination of binomials dealt with in the previous proposition. Each component standard deviation must, of course, be weighted with the total frequency of the component, and there must be the proper reduction to the mean of the array as a whole.

The $(i+1)^{\text{th}}$ binomial $(\frac{1}{2}u + \frac{1}{2}v)^{n-i}$ has $\sqrt{(n-i)\frac{1}{2}\times\frac{1}{2}}$[†] for its standard deviation, and the distance of its mean from the mean of the array

$$= \{n + 1 - (i + 1 + \tfrac{1}{2}(n - i))\} - \{\tfrac{1}{2}n - \tfrac{1}{3}(n - s)\}$$
$$= \tfrac{1}{3}(n - s) - \tfrac{1}{2}i.$$

Further, the frequency of this component is

$$= c_{n-s,\,i,\,o}\,2^i f_1.$$

We thus see that it contributes

$$c_{n-s,\,i,\,o}\,2^i f_1 \left\{\frac{n-i}{4} + \left(\frac{n-s}{3} - \frac{i}{2}\right)^2\right\}$$

to the total second moment about the mean of the array. This gives us

$$f_1 \times 3^{n-s} \times \sigma_s^2 = f_1 \Sigma c_{n-s,\,i,\,o}\,2^i \left\{\frac{n-i}{4} + \left(\frac{n-s}{3} - \frac{i}{2}\right)^2\right\},$$

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[*] "GALTON'S law makes the amount of inheritance an absolute constant for each pair of relatives. It would thus appear not to be a character of race or species, or one capable of modification by natural selection." More ample statistical experience of populations since 1885 shows that *absolute* constancy of the heredity coefficients is not consonant with actual measurements.—'Roy. Soc. Proc.,' vol. 62, p. 411.

[†] 'Phil. Trans.,' A, vol. 185, p. 373.

or

$$3^{n-s}\sigma_s^2 = \Sigma c_{n-s,\,i,\,o}\, 2^i \left\{\frac{n}{4} + \frac{(n-s)^2}{9} + \frac{i\,(i-1)}{4} - i\,\frac{n-s}{3}\right\}.$$

Now

$$(1 + 2x)^{n-s} = \Sigma c_{n-s,\,i,\,o}\,(2x)^i,$$

and by differentiating

$$2\,(n-s)\,(1+2x)^{n-s-1} = \Sigma c_{n-s,\,i,\,o}\,2^i i x^{i-1}.$$

Repeating the process

$$4\,(n-s)\,(n-s-1)\,(1+2x)^{n-s-2} = \Sigma c_{n-s,\,i,\,o}\,2^i i\,(i-1)\,x^{i-2}.$$

Hence, putting $x = 1$, we have the required expressions on the right of the above result, or

$$3^{n-s}\sigma_s^2 = 3^{n-s}\left\{\frac{n}{4} + \frac{(n-s)^2}{9} + \frac{(n-s)\,(n-s-1)}{9} - \frac{2\,(n-s)^2}{9}\right\}.$$

Therefore

$$\sigma_s^2 = \tfrac{1}{4}n - \tfrac{1}{9}\,(n-s) = \tfrac{1}{36}\,(5n + 4s).$$

Now the standard deviation of the whole population, as far as allogenic units are concerned, is

$$\sigma = \sqrt{n\tfrac{1}{4}\tfrac{3}{4}} = \sqrt{\tfrac{3}{16}n}.$$

Thus

$$\frac{\sigma_s}{\sigma} = \sqrt{\tfrac{4}{27}\left(5 + 4\,\frac{s}{n}\right)}.$$

This result is of singular interest. The variability of an array of offspring corresponding to a father of given allogenic constitution is not independent of the father, but increases steadily from a minimum of $\sqrt{\tfrac{5}{36}n}$, when there are no allogenic couplets in the father, to a maximum of $\sqrt{\tfrac{1}{4}n}$ when there are only allogenic couplets. In other words, fixing our attention on the same character, let the offspring of the hybrids *inter se* be crossed first with one pure race, and then with the second pure race, *i.e.*, first with pure allogenic and then with pure protogenic individuals, there ought to be a marked difference in the variability of the resulting offspring in the two cases.

Corollary (i.).—In the theory of *linear* regression as apart from the theory of normal correlation on the basis of the Gauss-Laplacian distribution,* if σ be the standard deviation of any character, and r its correlation with a second character, then

$$\sigma\sqrt{1 - r^2}$$

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* YULE, 'Journal of Royal Statistical Society,' vol. 60, December, 1897.

is the *mean* standard deviation of all the arrays of the first character for a given value of the second. This expression is no longer the actual standard deviation of each array.

It is of interest to see that this general law of linear regression is verified in the present case. We have $\sigma = \sqrt{\frac{3}{16}n}$ and $r = \frac{1}{3}$. Hence if Σ_m be the mean standard deviation of the arrays, we should expect

$$\Sigma_m^2 = \tfrac{3}{16}n\left(1 - \tfrac{1}{9}\right) = \tfrac{1}{6}n.$$

Remembering the weight of each array,* we have

$$\begin{aligned} 4^n \times 4^n \times 4^n \times \Sigma_m^2 &= \overset{s=n}{\underset{s=o}{\mathrm{S}}} \left\{ 4^n \times 4^n c_{n,s,o} 3^{n-s} \frac{5n+4s}{36} \right\} \\ &= 4^n \times 4^n (1+3)^n \tfrac{5}{36}n + 4^n \times 4^n n 4^{n-1} \tfrac{4}{36} \\ &= 4^n \times 4^n \times 4^n \times \tfrac{1}{6}n, \end{aligned}$$

whence $\Sigma_m^2 = \frac{1}{6}n$, as we anticipated.

Corollary (ii.).—It is clear that some arrays of offspring will be more, others less variable than the general population. The standard deviation of an array will be equal to that of the general population when s is found from

$$\tfrac{1}{36}(5n + 4s) = \tfrac{3}{16}n, \qquad \text{or} \qquad s = \tfrac{7}{16}n.$$

Now the mean number of allogenic couplets in the general population is $\frac{1}{4}n$. Thus the offspring array equally variable with the general population is at distance $\frac{3}{16}n$ from the mean. But σ for the general population $= \sqrt{\frac{3}{16}n}$. Hence, if we take fathers deviating from the population mean by $\sqrt{\frac{3}{16}n} \times \sigma$, we should expect their offspring to be equally variable with the general population. Supposing, therefore, the theory under discussion were true, we should have a means of finding, at least approximately, the number n of couplets corresponding to the character under consideration. All we should have to do would be to find the standard deviation of each array of offspring corresponding to a given father; these standard deviations ought to increase or decrease steadily across the table, their squares giving a straight line when plotted. Smooth the results, interpolate a value equal to σ, and we shall have the character of the father whose offspring are equally variable with the general population; but the deviation of this father from the mean ought to be $\sqrt{\frac{3}{16}n} \times \sigma$. Hence, since σ can be found, we have at once an approximate value of n. *Approximate* only, of course, because our arrays are classed by units of measurement, inches, centimetres, &c., and each such unit will not, as a rule, represent one allogenic couplet; but interpolation ought to give a result not widely divergent from the truth.

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* The number of offspring in the array due to the fathers with s-allogenic couplets is found at once by putting $u = v = w$ in the formula of Corollary (i.) on p. 61, and equals $4^n \times 4^n c_{n,s,o} 3^{n-s}$.

The method would of course fail practically if n were very large. For example, if n were 48, the deviation of the required group of fathers would be 3σ, and hence such a father would only occur once in 1000 individuals. In a manageable population, therefore, we are very unlikely to have enough such fathers to form any reliable measure of the variability of their offspring. At the same time, the squares of the variabilities of the arrays of sons due to quite frequent fathers ought to give a straight line, and if this line be determined properly, there should be no difficulty in finding the theoretical position of the above father, and so finding n.

Many other physiological theories besides the present might give this peculiarity of the diminishing variability of the arrays of offspring as we pass from one side of the correlation table to the other. Such changes in variability are familiar to those who have had to deal with skew correlation. But, as far as we are aware, they have not hitherto been noticed in inheritance tables. The existence of this changing variability would not affect in any way the general theory of linear regression applied to heredity in populations. It would, however, lead to an immediate extension of that theory consisting in the tabling of the standard deviations of the arrays. Should the standard deviations of these arrays show no bias towards a linear distribution, but only the fluctuation to be expected from random sampling about the mean value $\sigma\sqrt{(1 - r^2)}$, we should have a strong argument against the present general theory of alternative inheritance. We seem here, therefore, to have a crucial test of the validity of the theory, which may be quite as easy to apply as the previous test of the numerical value of the parental correlation.

Of course the results now reached are not consistent theoretically with normal correlation surfaces with their elliptic contour lines. The fact that Mr. GALTON came to his elliptic contours in the first instance on the basis of his observations, and not from any theory,* shows that they must in the case he was dealing with be approximately correct. Further, there is no doubt that in other statistics for characters in man there is within the limits of random sampling a close approximation to normal distribution. It might be hard to consider that such a deviation as would arise with a continuously increasing variability of the arrays from one side to the other of the table could exist and escape notice, had we not in physics had evidence that theory has often led to the discovery of an obvious relation which time after time must have been overlooked by previous observers unprovided with the theoretical hint of what to seek for. Hence while we may say that the parental correlation given by the theory is too rigid for the facts, we must leave this second test until more careful examination *ad hoc* has been made of the ample existing data.

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* 'Natural Inheritance,' p. 101.

(8.) PROPOSITION VI.—*To find the Array of Offspring due to a Grandfather of s-allogenic Couplets, supposing Complete Random Mating in the Population.*

The general distribution of the population is

$$(u + 2v + w)^n.$$

The fathers with s-allogenic units in their correlation are given by

$$c_{n,s,o}\, u^s\, (2v + w)^{n-s}.$$

The array of offspring due to these fathers is simply obtained by writing U for u, V for v, and W for w, and multiplying by $4^n \times 4^n$. This is a general rule for getting the offspring from any father if he mates at random. It gives us, as on p. 61, for the offspring distribution

$$4^n \times 4^n \times c_{n,s,o}\, \mathrm{U}^s\, (2\mathrm{V} + \mathrm{W})^{n-s}$$

$$= 4^n \times 4^n \times c_{n,s,o}\, (\tfrac{1}{2}u + \tfrac{1}{2}v)^s\, \{(\tfrac{1}{2}u + \tfrac{1}{2}v) + (v + w)\}^{n-s}.$$

To get the offspring of this array treated as fathers and mating at random, we have only to repeat the process, and we find

Offspring of grandfather of s-allogenic couplets

$$= 4^n \times 4^n \times 4^n \times 4^n c_{n,s,o}\, (\tfrac{1}{2}\mathrm{U} + \tfrac{1}{2}\mathrm{V})^s\, \{\tfrac{1}{2}\mathrm{U} + \tfrac{3}{2}\mathrm{V} + \mathrm{W}\}^{n-s}$$

$$= 4^{4n} c_{n,s,o}\, (\tfrac{3}{8}u + \tfrac{1}{2}v + \tfrac{1}{8}w)^s\, (\tfrac{3}{8}u + \tfrac{12}{8}v + \tfrac{7}{8}w)^{n-s}$$

$$= 4^{4n} c_{n,s,o}\, (\tfrac{3}{8}u + \tfrac{5}{8}\epsilon)^s\, (\tfrac{3}{8}u + \tfrac{5}{8}\epsilon + \tfrac{2}{5}(v + w))^{n-s} \times (\tfrac{5}{3})^{n-s},$$

where

$$\epsilon = \tfrac{4}{5}v + \tfrac{1}{5}w,$$

and is equal to unity if we identify v and w as something not allogenic. This can be dealt with exactly as in Proposition IV. we dealt with the array of offspring due to a father of s-allogenic couplets, *i.e.*, by analysing the array into the sum of a number of weighted binomials; in this case all skew.

Writing as before, $\eta = \frac{1}{2}(v + w)$, we have to expand

$$(\tfrac{3}{8}u + \tfrac{5}{8}\epsilon)^s\, (\tfrac{3}{8}u + \tfrac{5}{8}\epsilon + \tfrac{4}{5}\eta)^{n-s}.$$

The general term is

$$c_{n-s,i,o}\, (\tfrac{4}{5})^i \eta^i\, (\tfrac{3}{8}u + \tfrac{5}{8}\epsilon)^{n-i}.$$

This has a total frequency $c_{n-s,i,o}(\frac{4}{5})^i \times f_1$, and its mean is at a distance $i + 1 + \frac{5}{8}(n - i)$ from the origin which is taken at $(n + 1)$ allogenic couplets.

The total frequency of the array is $(1 + \frac{4}{5})^{n-s} f_1$. Hence, if m'_s be the mean of the grandchildren measured from the same origin, we have

$$f_1 \times (\tfrac{9}{5})^{n-s} \times m'_s = f_1 \{1 + \tfrac{5}{8}n + \tfrac{4}{5}(n-s)\{2 + \tfrac{5}{8}(n-1)\}$$

$$+ (\tfrac{4}{5})^2 \frac{(n-s)(n-s-1)}{1.2} \{3 + \tfrac{5}{8}(n-2)\} \ldots$$

$$+ (\tfrac{4}{5})^i \frac{(n-s)(n-s-1)\ldots(n-s-i+1)}{1.2.3\ldots i} (i + 1 + \tfrac{5}{8}(n-i)\}$$

$$= f_1 \{(\tfrac{5}{8}n + 1) \times (\tfrac{9}{5})^{n-s} + \tfrac{4 \cdot 3}{5 \cdot 8}(n-s)(\tfrac{9}{5})^{n-s-1}\},$$

or

$$m'_s = 1 + \tfrac{5}{8}n + \tfrac{4 \cdot 3 \cdot 5}{5 \cdot 8 \cdot 9}(n-s) = 1 + \tfrac{5}{8}n + \tfrac{1}{6}(n-s).$$

Thus

Mean of grandchildren $= \frac{3}{8}n - \frac{1}{6}(n-s)$.

Deviation from general population mean $= \frac{1}{8}n - \frac{1}{6}(n-s) = \frac{1}{24}(4s-n)$.

Deviation of grandparent from general population mean $= s - \frac{1}{4}n = \frac{1}{4}(4s-n)$.

Hence

$$\frac{\text{Deviation of offspring}}{\text{Deviation of grandparent}} = \tfrac{1}{6}.$$

This ratio is the same whatever be the allogenic constitution of the grandparent.

(9.) PROPOSITION VII.—*To find the Array of Offspring due to an m^{th} Great-grandfather of s-allogenic Couplets, supposing Complete Random Mating in each Generation.*

The array due to a father of s-allogenic couplets is

$$4^n \times 4^n \times c_{n,s,o} \{\tfrac{1}{2}(u+v)\}^s \{\tfrac{1}{2}(u+v) + (v+w)\}^{n-s},$$

and, as we have already seen, we must multiply by $4^n \times 4^n$ and put $\frac{1}{2}(u+v)$ for u, $\frac{1}{4}(u+2v+w)$ for v, and $\frac{1}{2}(v+w)$ for w to get the array due to the grandparent of s allogenic units. This process must be repeated m times if we wish to obtain the array due to the m^{th} great-grandparent.

We must first investigate what happens to $\frac{1}{2}(u+v)$ if this interchange be made m times. Suppose that it has been done i times, and let the answer be

$$M_i \tfrac{1}{2}(u+v) + M'_i \tfrac{1}{2}(v+w).$$

Repeat the operation, and the expression becomes

$$(\tfrac{3}{4}M_i + \tfrac{1}{4}M'_i)\tfrac{1}{2}(u+v) + (\tfrac{1}{4}M_i + \tfrac{3}{4}M'_i)\tfrac{1}{2}(v+w),$$

or

$$M_{i+1} = \tfrac{3}{4}M_i + \tfrac{1}{4}M'_i,$$

$$M'_{i+1} = \tfrac{1}{4}M_i + \tfrac{3}{4}M'_i.$$

Therefore

$$M_{i+1} + M'_{i+1} = M_i + M'_i = M_0 + M'_0 = 1 + 0 = 1.$$

Hence

$$M_{i+1} = \tfrac{1}{2}M_i + \tfrac{1}{4}, \quad M'_{i+1} = \tfrac{1}{2}M'_i + \tfrac{1}{4},$$

$$M_{i+1} - \tfrac{1}{2} = \tfrac{1}{2}(M_i - \tfrac{1}{2}) = \frac{1}{2^{i+1}}(M_0 - \tfrac{1}{2}) = \frac{1}{2^{i+2}},$$

$$M'_{i+1} - \tfrac{1}{2} = \tfrac{1}{2}(M'_i - \tfrac{1}{2}) = \frac{1}{2^{i+1}}(M'_0 - \tfrac{1}{2}) = -\frac{1}{2^{i+2}}.$$

Hence, finally,

$$M_i = \tfrac{1}{2} + \frac{1}{2^{i+1}}, \quad M'_i = \tfrac{1}{2} - \frac{1}{2^{i+1}}.$$

Thus the result of m changes on $\tfrac{1}{2}(u + v)$ is known.

Similarly the result of m changes on $\tfrac{1}{2}(v + w)$ is

$$M_i \frac{(v + w)}{2} + M'_i \frac{u + v}{2}.$$

We can now write down the array of offspring due to an m^{th} great-grandparent of s-allogenic couplets. It is

$$(4^n \times 4^n)^m c_{n,s,o}\left(M_m \frac{u + v}{2} + M'_m \frac{v + w}{2}\right)^s$$
$$\times \left\{(M_m + 2M'_m)\frac{u + v}{2} + (M'_m + 2M_m)\frac{v + w}{2}\right\}^{n-s}.$$

We must now find the mean of this array. For brevity let us write

$$M_m \tfrac{1}{2}(u + v) + M'_m \tfrac{1}{2}(v + w) = \mu u + \lambda\epsilon$$

$$(M_m + 2M'_m)\frac{u + v}{2} + (M'_m + 2M_m)\frac{v + w}{2} = \frac{M_m + 2M'_m}{M_m}(\mu u + \lambda\epsilon + \nu\eta),$$

where

$$\mu = \tfrac{1}{2}M_m, \quad \lambda = 1 - \tfrac{1}{2}M_m,$$

$$\epsilon = \frac{1}{1 + M'_m}v + \frac{M'_m}{1 + M'_m}w,$$

$$\nu = \frac{2(M_m^2 - M'^2_m)}{M_m + 2M'_m} = \frac{2(M_m - M'_m)}{1 + M'_m},$$

$$\eta = \tfrac{1}{2}(v + w)$$

Hence we have to find the mean of the system

$$(\mu u + \lambda\epsilon)^s (\mu u + \lambda\epsilon + \nu\eta)^{n-s}.$$

The i^{th} component binomial of this sum of binomials is

$$c_{n-s,\,i,\,o}\, \nu^i \eta^i (\mu u + \lambda\epsilon)^{n-i}.$$

It therefore has its mean at a distance

$$i + 1 + \lambda (n - i)$$

from $(n + 1)$ allogenic couplets, and a frequency given by

$$f_i = c_{n-s,\,i,\,o}\, \nu^i f_1.$$

The total frequency of the whole array $= (1 + \nu)^{n-s} f_1$.

Hence, taking moments round the origin at $n + 1$ allogenic couplets, we have, if m'_s be the mean of the array,

$$\begin{aligned}(1 + \nu)^{n-s} f_1 \times m'_s = f_1 \Big\{ & 1 + \lambda n + \nu (n - s)(2 + \lambda (n - 1)) \\ & + \nu^2 \frac{(n - s)(n - s - 1)}{1\,.\,2} (3 + \lambda (n - 2)) + \dots \\ & + \nu^i \frac{(n - s)(n - s - 1) \dots (n - s - i + 1)}{1\,.\,2\,.\,3 \dots i} (1 + i + \lambda (n - i)) + \dots \Big\}.\end{aligned}$$

Summing and dividing by $(1 + \nu)^{n-s}$ we find

$$m'_s = 1 + \lambda n + \frac{(n - s)\,\nu\,(1 - \lambda)}{1 + \nu}.$$

Hence the mean number of allogenic couplets in the members of the array

$$= n + 1 - m'_s = n (1 - \lambda) - (n - s) \frac{\nu}{1 + \nu} (1 - \lambda).$$

Deviation of offspring from mean of general population

$$= s \frac{\nu (1 - \lambda)}{1 + \nu} - n \left\{ \tfrac{1}{4} - \frac{1 - \lambda}{1 + \nu} \right\}.$$

We now substitute for ν and λ in terms of M_m and M'_m, and find

$$\frac{\nu (1 - \lambda)}{1 + \nu} = \tfrac{1}{3} (M_m - M'_m) = \tfrac{1}{3} \frac{1}{2^m}$$

$$\tfrac{1}{4} - \frac{1 - \lambda}{1 + \nu} = \tfrac{1}{4} - \tfrac{1}{6} (1 + M'_m) = \frac{1 - 2M'_m}{12} = \tfrac{1}{12} \frac{1}{2^m}.$$

Hence: Deviation of offspring $= \frac{1}{12}\frac{4s - n}{2^m}$; but the deviation of m^{th} great-grandparent $= s - \frac{1}{4}n$.

Thus we have

$$\frac{\text{Deviation of offspring}}{\text{Deviation of } m^{\text{th}} \text{ great-grandparent}} = \frac{1}{3}\frac{1}{2^m}.$$

This result is independent of s and of n.

Thus we conclude:

(i.) The regression of offspring on any individual ancestor is linear;

(ii.) The correlation coefficient is halved at each stage in ancestry;

(iii.) The result is perfectly independent of the number of couplets introduced into the formula.

The first two results are very familiar to biometric workers in heredity.

The actual numerical values of the grandparental, great-grandparental, great-great-grandparental correlations are $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{24}$, &c.

These are distinctly less than the values so far reached for ancestral correlation, the grandparental correlations, for instance, lying between ·2 and ·3.

The results show, however, that a general theory of the pure gamete, embracing the simpler forms of the Mendelian principle, leads us directly to a series of ancestral correlations decreasing in a geometrical progression. Thus, when we suppose a population arising from hybridisation to cross at random, we find that it obeys the second fundamental assumption of the biometric theory of heredity.* In other words, ancestry is of the utmost importance, and the population follows laws identical in form with those propounded in the biometrical theory on the basis of a linear regression multiple correlation. Only the values of the constants deduced for the law of ancestral heredity from the present theory of the pure gamete (which appears to cover the bulk of Mendelian formulæ hitherto propounded) are sensibly too small to satisfy the best recent observations on inheritance.

It is of interest to find "Mendelian Principles" when given a wide analytical expression leading up to the very laws of linear regression, of distribution of frequency, and of ancestral inheritance in populations, which have been called into question as exhibiting only a blurred and confused picture of what actually takes place.

It would be an immense advantage if we could accept such a theory of the pure gamete as has been here analysed as a physiological basis for the theory of heredity. We should then have a physiological origin for the ideas of regression and of ancestral inheritance which statistics of heredity in populations have made familiar to biometric workers. Unfortunately, even such a general pure gamete theory as we have here dealt with, while leading to results which form a special case of the law of ancestral heredity, is not sufficiently elastic to cover the observed facts. The lesson to be learnt from the present investigation is, however, that there is no essential repugnance between any of the main results of the biometric school and a theory of the pure gamete, but, on the contrary, it is perfectly possible to test such theories by biometric methods. We may fairly ask anyone who propounds in future a Mendelian or pure gamete formula as a general theory of heredity, to remember that it involves in itself definite laws regulating the reproduction of a population mating at random, and that it is incumbent on the propounder to test whether or not such laws are consistent with what we already know of the inheritance statistics of such populations. When we remember that deducing all the effects of such a formula within the whole field of inheritance will almost always form a very laborious piece of mathematical analysis, there seems a touch of scientific irresponsibility in propounding an immense variety of formulæ to suit one or other special case, and the modifying or withdrawing them when they are found to fail in another.

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* 'Biometrika,' vol. 2, p. 220.

(10.) PROPOSITION VIII.—*To find the Regression and Correlation of Brethren on the Theory developed in this Paper.*

We shall suppose the group of brethren to consist of 4χ members, or any pair of parents to have a family of 4χ.

Consider first parents of *one couplet only*, the offspring of the 16 possible pairs are given in the table below :—

		Father.			
		$a+a$.	$a+A$.	$A+a$.	$A+A$.
Mother	$a+a$	$4\chi u$	$2\chi(u+v)$	$2\chi(v+u)$	$4\chi v$
	$a+A$	$2\chi(u+v)$	$\chi(u+2v+w)$	$\chi(u+2v+w)$	$2\chi(v+w)$
	$A+a$	$2\chi(v+u)$	$\chi(u+2v+w)$	$\chi(u+2v+w)$	$2\chi(w+v)$
	$A+A$	$4\chi v$	$2\chi(v+w)$	$2\chi(w+v)$	$4\chi w$

Now let us take every pair of brothers out of each of these 16 families and form a correlation table of brothers. The following table gives the distribution of the various types :—

		First brother.		
		u.	v.	w.
Second brother	u	$4\chi(9\chi-4)$	$24\chi^2$	$4\chi^2$
	v	$24\chi^2$	$80\chi^2-32\chi$	$24\chi^2$
	w	$4\chi^2$	$24\chi^2$	$4\chi(9\chi-4)$

This gives a total of $256\chi^2 - 64\chi = 16 \times 4\chi(4\chi - 1)$ pairs of brothers, as it should, every brother in 16 families having $4\chi - 1$ brethren, and there being 4χ in each family. Clearly we can divide by 4χ, and we have the simplified form :—

	u_1.	v_1.	w_1.
u_2	$9\chi - 4$	6χ	χ
v_2	6χ	$20\chi - 8$	6χ
w_2	χ	6χ	$9\chi - 4$

where the subscripts 1 and 2 refer to the first and second brothers.

Let us simplify this by considering only the allogenic elements, η denoting either a heterogenic or a protogenic element. Then we have :—

	u_1.	η_1.
u_2	$9\chi - 4$	7χ
η_2	7χ	$41\chi - 12$

Thus the distribution of pairs of brothers in the case of the character being fixed by a single couplet is

$$(9\chi - 4)\,u_1u_2 + 7\chi u_1\eta_2 + 7\chi u_3\eta_1 + (41\chi - 12)\,\eta_1\eta_2.$$

This is to be read as follows: there are $9\chi - 4$ cases of both brothers being allogenic, to $41\chi - 12$ cases of neither brother allogenic, to $7\chi + 7\chi$ cases of one only of the two brothers being allogenic.

Now when we pass from a character fixed by one couplet to a character fixed by two, the above distribution can occur in either couplet, and every possible pair of brothers will be got by squaring the above expression. Proceeding in this way to n couplet characters, we have the following symbolic expression for the distribution of brothers

$$(4\chi)^n \times \{9\chi - 4)\,u_1u_2 + 7\chi u_1\eta_2 + 7\chi u_2\eta_1 + (41\chi - 12)\,\eta_1\eta_2\}^n,$$

the omitted factor 4χ being restored.

This, I think, represents the distribution of a character measured by allogenic couplets in a population of pairs of brothers, *i.e.*, is the correlation table for brothers in the population—any term involving $u_1^p u_2^q$ representing when we put η_1 and η_2 equal to unity the number of pairs in which the first brother has p and the second q allogenic couplets in their constitutions.

Hence if we find the coefficient of u_1^p in terms of u_2, we shall have the array or brothers to be found associated with a brother of p allogenic couplets. The above expression may be written

$$(4\chi)^n [\{(9\chi - 4) u_2 + 7\chi\eta_1\} u_1 + \{7\chi u_2 + (41\chi - 12) \eta_1\} \eta_2]^n.$$

The term involving $u_1{}^p$ is

$$(4\chi)^n \{(9\chi - 4) u_2 + 7\chi\eta_1\}^p \{7\chi u_2 + (41\chi - 12) \eta_1\}^{n-p} \eta_2{}^{n-p} c_{n, p, o}.$$

Neglecting the constant factor, the distribution in u_2 is given by

$$\{(9\chi - 4) u_2 + 7\chi\eta_1\}^p \{7\chi u_2 + (41\chi - 12) \eta_1\}^{n-p}.$$

We require to find the mean of this array.

Put

$$\lambda = \frac{9\chi - 4}{16\chi - 4}, \qquad \mu = \frac{7\chi}{16\chi - 4}, \qquad \nu = \frac{4(5\chi - 3)}{7\chi}.$$

Then again, but for a factor independent of the power of u_2, the array may be read

$$(\lambda u_2 + \mu\eta_1)^p (\lambda u_2 + \mu\eta_1 + \nu\eta_1)^{n-p}.$$

The general term is therefore

$$(\lambda u_2 + \mu\eta_1)^{p+s} (\nu\eta_1)^{n-p-s} c_{n-p, s, o},$$

and we must sum from $s = o$, to $s = n - p$.

The general term has therefore its mean at the distance $1 + \mu(p + s)$ from $p + s + 1$ allogenic couplets, or its mean

$$= (1 - \mu)(p + s), \text{ allogenic couplets,}$$

and its total frequency $= \nu^{n-p-s} c_{n-p, s, o}$.

This gives a total frequency of the array proportional to $(1 + \nu)^{n-p}$.

Hence, taking moments, we have for the mean m of the array given by

$$m \times (1 + \nu)^{n-p} = \mathop{\mathrm{S}}_{s=o}^{s=n-p} \{(1 - \mu)(p + s) \nu^{n-p-s} c_{n-p, s, o}\}$$
$$= (1 - \mu) p (1 + \nu)^{n-p} + (1 - \mu)(n - p)(1 + \nu)^{n-p-1}.$$

Therefore

$$m = \frac{1 - \mu}{1 + \nu} n + \frac{\nu(1 - \mu)}{1 + \nu} p.$$

Hence we see that

(i.) The regression between brothers is linear.

(ii.) The fraternal correlation which is equal to the regression =

$$\frac{(5\chi - 3)}{3(4\chi - 1)},$$

and is quite independent of the number of couplets. It is, however, a function of χ, the size of the family used in forming the table. We have the following values :—

Size of family.	Value of χ.	Value of fraternal correlation.
8	$\chi = 2$	·3333
12	3	·3636
16	4	·3778
20	5	·3860
24	6	·3913
32	8	·3978
40	10	·4017
∞	∞	·4067

The value of fraternal correlation thus varies with the size of the family dealt with from ·3 to ·4. Probably the more correct way of looking at any fraternal correlation table would be to suppose it a random sample of all the pairs of brothers which would be obtained by giving a large, or even indefinitely large, fertility to each pair, for what we actually do is to take families of varying size and take as many pairs of brethren as they provide. In this case we ought to reach a fraternal correlation of ·4, precisely the value reached by the ancestral law when we take FRANCIS GALTON'S original series of ancestral correlations.*

Thus we conclude that on the general theory of the pure gamete here dealt with, the fraternal correlation is slightly larger than the parental. This is in accordance with the general result of biometric investigations on populations. But the value, as in the case of the parental correlation, is very sensibly lower than the value—about ·5—found from recent investigations on man. It is further very inelastic even if we allow for some variation in the size of families dealt with. There can be little doubt that fraternal correlation varies from character to character and species to species in a manner sensibly beyond what can be accounted for by differences in the size of the family dealt with.†

Corollary.—We can exhibit the regression in the form:

Mean of array − mean of general population

$$= \frac{\nu(1-\mu)}{1+\nu}\{\text{deviation of brother from mean of general population}\},$$

by observing that $1 - \mu = \frac{1}{4}(1 + \nu\mu)$, whence

$$m = \tfrac{1}{4}\frac{(1+\nu\mu)}{1+\nu}n + \frac{\nu(1-\mu)}{1+\nu}p,$$

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* 'Roy. Soc. Proc.,' vol. 62, p. 410.

† There is sensible variation even for different characters, when we take the same series of pairs of brothers, and only one pair from each family.

or

$$m - \tfrac{1}{4}n = \frac{\nu(1-\mu)}{1+\nu}(p - \tfrac{1}{4}n).$$

(11.) PROPOSITION IX.—*To find the General Formula for Biparental Regression on the Theory of the Pure Gamete, and the Value to be given to the "Midparent."*

If we applied without further consideration the general formula for biparental regression to this case, we should have, if m_{pq} be the mean of the offspring due to fathers of p-allogenic couplets, mated with mothers of q-allogenic couplets,

$$m_{pq} = \tfrac{1}{4}n + \tfrac{1}{3}(p - \tfrac{1}{4}n) + \tfrac{1}{3}(q - \tfrac{1}{4}n).$$

This follows at once, since the mean of the general population $= \frac{1}{4}n$, the regression coefficient for either parent $= \frac{1}{3}$, and there is no assortative mating.

Hence we should have

$$m_{pq} = \tfrac{1}{12}n + \tfrac{1}{3}(p + q).$$

Now suppose both parents of pure allogenic race, then $p = q = n$, and all the offspring will be of pure allogenic race, or we must have $m_{pq} = n$.

But the above formula gives

$$m_{pq} = \tfrac{3}{4}n,$$

which is not correct.

In other words, while the above formula gives the best plane to fit the array of points determined by the parental constitutions, that array of points does not truly lie in a plane. Or, although the simple regressions are linear, the compound regression is not truly planar. We have therefore to find its true form, and measure the amount of deviation from the truth involved in using a biparental formula of the type indicated.

Given a character resulting from n couplets, we require $4^n \times 4^n$ individuals, 4^n male and 4^n female, to form the whole possible system of random matings. In such a population there would be

$$c_{n,p,0}3^{n-p} \text{ fathers of } p\text{-allogenic couplets,}$$

and

$$c_{n,q,0}3^{n-q} \text{ mothers of } q\text{-allogenic couplets.}$$

The chance therefore of a mating of a p-allogenic father and a q-allogenic mother is

$$c_{n,p,0}\, c_{n,q,0}\, 3^{2n-p-q}/4^{2n},$$

and when n be even moderately large, this gets very small if p and q at all nearly approach n. For example, if n were 5, and father and mother were both pure allogenes, the chance of such a pure allogenic mating would be only

$$1/1,048,576,$$

or in a population of a million would hardly occur once.

Still keeping $n = 5$, take $p = q = \frac{4}{5}n = 4$. Then we have the chance of such a mating

$$= 1/4215$$

still extremely improbable.

Thus when n is even moderately large, pure allogenic matings are so rare that they have vanishingly small influence in the population at large. Even if n were 2, the chance of a pure allogenic mating is only $\frac{1}{256}$. These points must be borne in mind in what follows.

Consider first a father and a mother of one couplet each, their zygotes are either u, v or w, involving a gametic constitution $a + a$, $a + A$ or $A + A$. We have the following scheme :—

Zygote of father.	Zygote of mother.	Number of matings.	Offspring.
u	u	1	$4u$
u	v	2	$2(u + v)$
u	w	1	$4v$
v	u	2	$2(v + u)$
v	v	4	$u + 2v + w$
v	w	2	$2(w + v)$
w	u	1	$2(v + v)$
w	v	2	$2(w + v)$
w	w	1	$4w$

Hence it there be

1 allogenic couplet in father and 1 in mother, offspring $= 4u$,
1 . . . . 0 . . . . $= 4(u + 2v)$.
0 . . . . 1 . . . . $= 4(2v + u)$,
0 . . . . 0 . . . . $= 4(u + 4v + 4w)$.

Let us write

$$16j_0 = 16(\tfrac{1}{4}u + v + w), \qquad 16j_1 = 16(\tfrac{1}{4}u + \tfrac{1}{2}v), \qquad 16j_2 = 16\tfrac{1}{4}u.$$

Then consider the relation

$$(u^0\epsilon^0 + u'\epsilon^1) \times (u'^0\eta^0 + u'^1\eta^1) = 4^2(j_0\epsilon^0\eta^0 + j_1(\epsilon^1\eta^0 + \epsilon^0\eta^1) + j_2\epsilon^1\eta^1),$$

where ϵ and η are mere symbols, and 0, 1, etc., denote their powers. u, u' refer respectively to father and mother, and their powers denote the number of allogenic couplets in the zygotes of father and mother. Then the above is a symbolical relation which gives, by equating any power or product of ϵ and η on either side, the offspring of a pair of parents of definite constitution.

Now suppose the parents not to consist of a single couplet, but of n couplets, then the total distribution of offspring that we have given above for any couplet may occur in each couplet, and each such distribution must be combined with every other couplet distribution. We then reach, dropping unnecessary indices, the general symbolic relation.

$$\begin{aligned}&(u^0 + u^1\epsilon + u^2\epsilon^2 + \dots + u^n\epsilon^n)\\ &\times (u'^0 + u'^1\eta + u'^2\eta^2 + \dots + u'^n\eta^n)\\ &= 4^n \times 4^n \times (j_0 + j_1(\epsilon + \eta) + j_2\epsilon\eta)^n.\end{aligned}$$

Thus the array of offspring due to parents of zygotes with p and q allogenic couplets respectively—*i.e.*, to $u^p \times u'^q$ is the coefficient of $\epsilon^p\eta^q$ on the right-hand side, or in the expansion of

$$4^n \times 4^n \times (j_0 + j_1(\epsilon + \eta) + j_2\epsilon\eta)^n.$$

This may be written

$$4^n \times 4^n \times (j_1 + j_2\eta)\epsilon + j_0 + j_1\eta)^n.$$

Thus the coefficient of ϵ^p is

$$4^n \times 4^n \times (j_1 + j_2\eta)^p (j_0 + j_1\eta)^{n-p} c_{n,p,0}.$$

We require to pick the coefficient of η^q out of this in order to get the array of offspring due to fathers of p, and to mothers of q, allogenic couplets. But this is clearly

$$\begin{aligned}4^n \times 4^n \times c_{n,p,0}\{&j_2^q j_1^{p-q} j_0^{n-p} c_{p,q,0}\\ &+ j_2^{q-1} j_1^{p-q+2} j_0^{n-p-1} (n-p) c_{p,q-1,0}\\ &+ j_2^{q-2} j_1^{p-q+4} j_0^{n-p-2} c_{n-p,2,0} c_{p,q-2,0}\\ &+ \dots\},\end{aligned}$$

or, more briefly,

$$4^n \times 4^n \times c_{n,p,0}\left\{\overset{r=q}{\underset{r=0}{S}} (j_2^{q-r} j_1^{p-q+2r} j_0^{n-p-r} c_{n-p,r,0} c_{p,q-r,0})\right\}.$$

We shall first find the mean and frequency corresponding to the r^{th} term as given above of this series. What we have to deal with is

$$(\tfrac{1}{4}u)^{q-r} (\tfrac{1}{4}u + \tfrac{1}{2}v)^{p-q+2r} (\tfrac{1}{4}u + v + w)^{n-p-r}.$$

We may write this

$$\frac{3^{n-q+r} \times u^{q-r}}{4^n} \times (\tfrac{1}{3}u + \tfrac{2}{3}\chi)^{p-q+2r} \times (\tfrac{1}{3}u + \tfrac{2}{3}v + 2\chi')^{n-p-r},$$

where $\chi = v$ and $\chi' = \frac{1}{3}v + \frac{2}{3}w$, and both may be put unity when we are merely finding the distribution of allogenic couplets.

Now the general term in the above expression is

$$\frac{3^{n-q+r} \times u^{q-r}}{4^n} \times (\tfrac{1}{3}u + \tfrac{2}{3}\chi)^{p-q+2r+s} (2\chi')^{n-p-r-s} c_{n-p-r,s,0}$$

and s must be taken from o to $n - p - r$. The frequency of this term is

$$\frac{3^{n-q+r}}{4^n} \, 2^{n-p-r-s} c_{n-p-r,\, s,\, o},$$

and its mean is $q - r + \frac{1}{3}(p - q + 2r + s)$ allogenic couplets.

The total frequency of the r^{th} term is therefore

$$\frac{3^{n-q+r}}{4^n}(1+2)^{n-p-r} = \frac{3^{2n-p-q}}{4^n},$$

which of course must ultimately be multiplied by the factorials in r omitted above. If m_r be the mean number of allogenic couplets in the r^{th} term, we have

$$\begin{aligned} m_r \times \frac{3^{2n-p-q}}{4^n} &= \overset{s=n-p-r}{\underset{s=o}{\mathrm{S}}} \Bigg(\frac{3^{n-q+r}}{4^n} 2^{n-p-r-s} c_{n-p-r,\, s,\, o} \{q - r + \tfrac{1}{3}(p - q + 2r + s)\} \\ &= \{q - r + \tfrac{1}{3}(p - q + 2r)\} \frac{3^{2n-p-q}}{4^n} + \tfrac{1}{3}(n - p - r) \frac{3^{2n-p-q-1}}{4^n} \Bigg). \end{aligned}$$

Thus:

$$\begin{aligned} m_r &= q - r + \tfrac{1}{3}(p - q + 2r) + \tfrac{1}{9}(n - p - r), \\ &= \tfrac{1}{9}n + \tfrac{2}{3}q + \tfrac{2}{9}p - \tfrac{4}{9}r. \end{aligned}$$

This is the mean of the r^{th} term, and its total frequency is

$$4^n \times 4^n \times \frac{3^{2n-p-q}}{4^n} c_{n,\, p,\, r} \times c_{p,\, q-r,\, o}.$$

Hence, if

$$\begin{aligned} \mathrm{F} &= 4^n \times 3^{2n-p-q} \overset{r=q}{\underset{r=o}{\mathrm{S}}} c_{n,\, p,\, r} \times c_{p,\, q-r,\, o} \\ &= 4^n 3^{2n-p-q} \times f, \end{aligned}$$

we shall have

$$f \times m_{pq} = \overset{r=q}{\underset{r=o}{\mathrm{S}}} \{(\tfrac{1}{9}n + \tfrac{2}{3}q + \tfrac{2}{9}p) - \tfrac{4}{9}r\} c_{n,\, p,\, r} \times c_{p,\, q-r,\, o},$$

where m_{pq} is the mean of the array of offspring due to fathers of p and mothers of q allogenic couplets.

The only difficulty here is summing the series

$$\overset{r=q}{\underset{r=o}{\mathrm{S}}} \{r c_{n,\, p,\, r} \times c_{p,\, q-r,\, o}\}.$$

But this may be written

$$\overset{r-1=q-1}{\underset{r-1=o}{\mathrm{S}}} (n \times c_{n-1,\, p,\, r-1} \times c_{p,\, q-1-r+1,\, o}),$$

or multipled by $4^{n-1}3^{2(n-1)-p-(q-1)}/n$ it represents the total offspring of fathers of p and mothers of $q - 1$ allogenic units in a population with $(n - 1)$ couplets in their constitution

$$= 4^{n-1} \times c_{n-1,\,p,\,o} \times c_{n-1,\,q-1,\,o}\, 3^{2(n-1)-p-q-1}.$$

Hence

$$\overset{n-1-q-1}{\underset{s=1}{S}} \{n \times c_{n-1,\,p,\,s-1} \times c_{p,\,q-1-s+1,\,o}\} = n \times c_{n-1,\,p,\,o} \times c_{n-1,\,q-1,\,o}.$$

Now

$$4^n \times 3^{2n-p-q} \times f = 4^n \times c_{n,\,p,\,o} \times c_{n,\,q,\,o}\, 3^{2n-p-q}.$$

Thus

$$f = c_{n,\,p,\,o} \times c_{n,\,q,\,o},$$

or, if f' denote the above series, we have

$$f' = \frac{q(n-p)}{n} f.$$

This leads us to

$$m_{pq} = \tfrac{1}{9}n + \tfrac{2}{3}q + \tfrac{2}{9}p - \tfrac{4}{9}\frac{q(n-p)}{n}$$

$$= \tfrac{1}{9}n + \tfrac{2}{9}(p+q) + \tfrac{4}{9}\frac{pq}{n}$$

$$= \tfrac{1}{9}\frac{(n+2p)(n+2q)}{n}.$$

This is a most remarkable result, for it shows that the regression surface is not a plane but a hyperboloid. Let us measure all the quantities in deviations from the mean of the general population, *i.e.*, put $m_{pq} = m'_{pq} + \frac{1}{4}n$, $p = p' + \frac{1}{4}n$, $q = q' + \frac{1}{4}n$. We find

$$m'_{pq} = \tfrac{1}{3}(p' + q') + \tfrac{4}{9}\frac{p'q'}{n}.$$

This formula reconciles at once the Mendelian and Galtonian positions. When the number of couplets is large, parents having a number of allogenic couplets comparable with n are vanishingly small in number. The standard deviation σ of the population is $\sqrt{3n}/4$ (see p. 57).

Hence we may write

$$m'_{pq} = \tfrac{2}{3}\left(\frac{p' + q'}{2}\right) + \frac{p'}{3\sigma}\frac{q'}{\sqrt{3n}}.$$

But p' can only once per thousand cases be as big as 3σ, and accordingly when n is large, both terms of the product will be small. In this case the surface becomes practically planar, and we have the Galtonian result of 1885,* that the offspring from the Galtonian "midparent" are one-third nearer the general mean of the population.

On the other hand, when n is small we see that for midparents not differing too widely from the population mean, Galtonian regression of the value $\frac{1}{3}$ holds, but that as we pass away from the mean of the population the regression gets less and less, becoming absolutely zero when we take two pure allogenic parents. Just as the regression is reduced when we emphasise the allogenic constitution of the parents, it is increased when we emphasise the non-allogenic elements.

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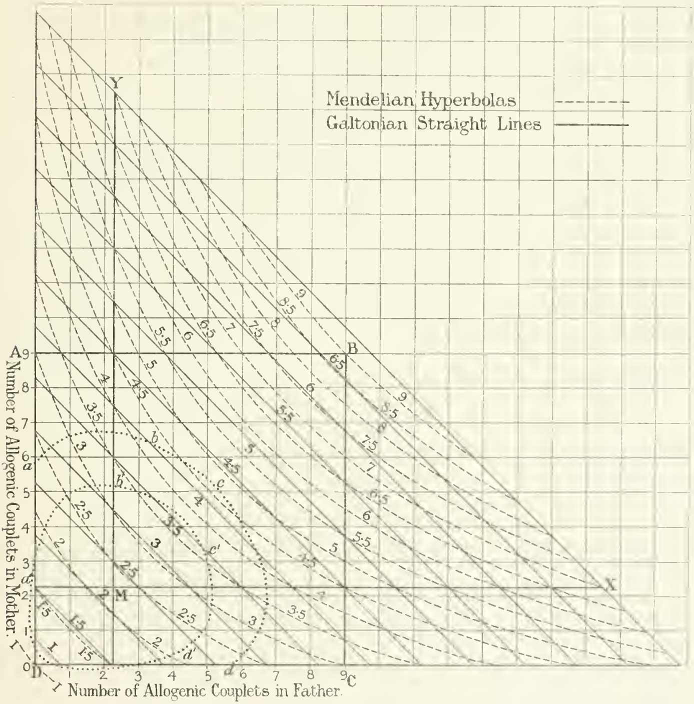
* 'Natural Inheritance,' p. 97.

To illustrate these points I have drawn a diagram for $n = 9$ of the contour lines of



Midparental contours.

the regression surface as a plane and a hyperboloid. The area ABCD contains all possible parents; 999 out of 1000 random matings fall in the loop *abcd*; 99 out of 100 random mating within the loop *a'b'c'd'*, which has been carried right round to mark that it excludes matings in which both parents have no allogenic couplets. The means of the array of offspring due to any pair of parents are marked in slim figures along the Mendelian hyperbolic contours, and in heavy figures along the Galtonian straight lines. We see comparatively small differences as long as we deal with matings within the 99 per cent. loop, somewhat greater differences as we approach the 99·9 per cent. loop, and very marked differences as we go beyond the latter boundary towards pure allogenic parentage. A study of the diagram illustrates at once how the Mendelian theory exhibits for the bulk of the population Galtonian regression, such regression becoming, however, less and less as we proceed to individuals, the frequency of whose matings may be less than one in a million.

It will be seen again that in this proposition we have no fundamental antagonism between the Mendelian and biometric standpoints. We reach a single formula which approaches more and more closely to the biometric standpoint when we deal with characters depending on many allogenic couplets.* On the other hand, it gives the absence of regression which is obtained when pure allogenic parents are mated. We see, however, quite clearly that it is totally erroneous to argue from this single case against regression in general. Such regression actually exists on "Mendelian Principles" when any population breeding at random is taken, and involves in itself the whole conception of ancestral correlation and the influence of ancestry.

Of the formula for the midparent now reached, however, we can only say that on the basis of our experience in populations the factor $\frac{2}{3}$ seems too inelastic to work. Here again the data must be especially investigated from the standpoint of a midparent given by

$$\tfrac{1}{2}(p' + q') + \tfrac{2}{3}\frac{p'q'}{n}$$

before judgment can be final on this test.

Theoretically, by assuming the midparent to be

$$\tfrac{1}{2}(p' + q') + \chi p'q',$$

we should have a means of finding χ by averages, and therefore n, the number of couplets involved.

(12.) *General Conclusions.*

In this paper we have dealt with a general theory of the pure gamete—possibly not the widest that could be conceived—but sufficiently wide to indicate the real bearing of Mendelian formulæ when applied to a population mating at random. We see that under such circumstances:

(i.) The population which results from the offspring of hybrids remains stable; every variation which appears, appears with a certain definite and predicable frequency; there is no room for the appearance of "mutations," although certain variations with very small frequency would be extremely rare in a limited population. A mutation—a variation not hitherto observed—would only appear in the offspring of the hybrids between two pure races; after this with random mating the mixed race would remain perfectly stable until disturbed by sexual or natural selection. These are the only mutations which arise on the generalised theory of the pure gamete, *i.e.*, *two* pure races form *one* mixed race, breeding true to itself; it is difficult under these circumstances to account for the origin of the two pure races by a mutation-theory of the differentiation of species!

---

* For $n = 50$ or 100 the population within the 1 in 1000 line sensibly obeys ordinary linear midparental regression.

(ii.) Between any two relations—if we measure the character by the number of allogenic or protogenic couplets in the zygote of the individual—we have a linear regression. The frequency distribution of any character is skew, approaching closely the normal distribution as the number of couplets which determines the constitution of the zygote is increased.

(iii.) The correlations between pairs of blood relations take definite numerical values absolutely independent of the number of couplets, and the same for all characters and races.

(iv.) The ancestral correlations form a geometrical series of common ratio one-half.

(v.) Fraternal correlation is fixed between narrow limits depending on the number of brothers per family dealt with, and is very slightly larger than parental correlation.

(vi.) The theory of the midparent for a considerable number of couplets approaches closely that originally given by FRANCIS GALTON, except for extreme values of the character, when the regression becomes rapidly smaller and ultimately vanishes.

We thus see that a generalised theory of the pure gamete would be of very great advantage if it could be accepted. It would lead to a system of inheritance in randomly mating populations with non-differential fertility, which in its broad features would be essentially the same as that which has been biometrically developed not from theoretical hypotheses, but from the statistical description of observed facts in populations.

Unfortunately, however, when we come to the actual numerical values for the coefficients of heredity deducible from such a theory of the pure gamete, they do not accord with observation. They diverge in two ways: First, they give a rigid value for these coefficients for all races and characters—a result not in reasonable accordance with observation. Secondly, they give values distinctly too small, as compared with the average values, or with the modal values of large series of population observations.

We thus reach the point we have so often had to insist upon: that the biometric or statistical theory of heredity does not involve a denial of any physiological theory of heredity, but it serves in itself to confirm or refute such a theory. Mendelian formulæ analytically developed for randomly mating populations are either consistent or not with the biometric observations on such populations. If they are consistent, it shows their possibility, but does not prove their necessity. If they are not, it shows they are inadequate. The present investigation shows that in the theory of the pure gamete there is nothing in essential opposition to the broad features of linear regression, skew distribution, the geometric law of ancestral correlation, etc., of the biometric description of inheritance in populations. But it does show that the generalised theory here dealt with is not elastic enough to account for the numerical values of the constants of heredity hitherto observed.

It will be time enough to consider other more or less general Mendelian formulæ when there is far better evidence than exists at present that they cover a real range of observation, and have not been *solely* invented to describe isolated experiences, the numerical results of which are not in complete accordance with simple Mendelianism. Given such neo-Mendelian formulæ, there is a perfectly straightforward mathematical method of applying them to randomly mating populations, but that method is excessively laborious, and the biometrician may well hesitate to undertake the task of their investigation. A few minutes suffice to invent a Mendelian formula, but weeks of labour may be involved in testing whether it leads to legitimate results when applied to sexually crossing races. Let us therefore have a few simple general principles stated which embrace *all* the facts deducible from the hybridisation experiments of the Mendelians; these can form the basis of a new mathematical investigation, but it is idle to undertake such an investigation so long as Mendelian Principles remain in a state of flux.

Any combination of the theory of pure gametes here discussed with homogamy, or with fertility correlated with homogamy, or again with prepotency of individual or of type, would emphasise the correlations which we have found above to be too low; but such hypotheses would involve a fundamental alteration in the formula

$$(a + a')(\mathrm{A} + \mathrm{A}') = a\mathrm{A} + a\mathrm{A}' + a'\mathrm{A}' + \mathrm{A}\mathrm{A}'.$$

Such a formula would then give the *possibilities* of the cross, but the proportions of these possibilities actually occurring would be quite different.*

Such loading of the possibilities—not only of the individual couplet—but very probably of associated couplets in the constitution—might conceivably enable us to deduce better values for the ancestral and collateral correlations. But it would abolish not only the simplicity of the fundamental Mendelian formula, it would also involve lengthy preliminary studies on homogamy, fertility, and prepotency before any effective formula could be propounded.

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* Toss two pennies, and the result of $4n$ tossings will closely approximate to the distribution $n\,(\mathrm{HH} + 2\mathrm{HT} + \mathrm{TT})$. Load one or both coins, and the possible variations will still be HH, HT or TT, but their proportions will be far from $n : 2n : n$.

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Phil. Trans., A, vol. 203, 1904, pp. 87–110.

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IV. *On the Acoustic Shadow of a Sphere.*

By Lord RAYLEIGH, *O.M., F.R.S.*

With an Appendix, giving the Values of LEGENDRE'S *Functions from* P_0 *to* P_{20} *at Intervals of 5 degrees. By Professor* A. LODGE.

Received December 28, 1903,—Read January 21, 1904.

IN my book on the 'Theory of Sound,' § 328, I have discussed the effect upon a source of sound of a rigid sphere whose surface is close to the source.

The question turns upon the relative magnitudes of the wave-length (λ) and the radius (c) of the sphere. If kc be *small*, where $k = 2\pi/\lambda$, the presence of the sphere has but little effect upon the sound to be perceived at a distance.

The following table was given, showing the effect in three principal directions of somewhat larger spheres :—

kc.	μ.	$F^2 + G^2$.
$\frac{1}{2}$	1	·294291
	−1	·259729
	0	·231999
1	1	·502961
	−1	·285220
	0	·236828
2	1	·6898
	−1	·3182
	0	·3562

Here $F^2 + G^2$ represents the intensity of sound at a great distance from the sphere in directions such that μ is the cosine of the angle between them and that radius which passes through the source. Upon the scale of measurement adopted, $F^2 + G^2 = \frac{1}{4}$ for all values of μ, when $kc = 0$, that is, when the propagation is undisturbed by any obstacle. The increased values under $\mu = 1$ show that the sphere is beginning to act as a *reflector*, the intensity in this direction being already more than doubled when $kc = 2$. "In looking at these figures, the first point which attracts attention is the comparatively slight deviation from uniformity in the intensities in different directions. Even when the circumference of the sphere amounts to twice the wave-length, there is scarcely anything to be called a sound shadow. But what is, perhaps, still more unexpected is that in the first two cases the intensity behind the sphere [$\mu = -1$] exceeds that in a transverse direction [$\mu = 0$]. This result depends mainly on the preponderance of the term of the first order, which vanishes with μ. The order of the more important terms increases with kc; when kc is 2, the principal term is of the second order.

"Up to a certain point the augmentation of the sphere will increase the total energy emitted, because a simple source emits twice as much energy when close to a rigid plane as when entirely in the open. Within the limits of the table this effect masks the obstruction due to an increasing sphere, so that when $\mu = -1$, the intensity is greater when the circumference is twice the wave-length than when it is half the wave-length, the source itself remaining constant."

The solution of the problem when kc is very great cannot be obtained by this method, but it is to be expected that when $\mu = 1$ the intensity will be quadrupled, as when the sphere becomes a plane, and that when μ is negative the intensity will tend to vanish. It is of interest to trace somewhat more closely the approach to this state of things—to treat, for example, the case of $kc = 10$.* In every case where it can be carried out the solution has a double interest, since in virtue of the law of reciprocity it applies when the source and point of observation are interchanged, thus giving the intensity at a point on the sphere due to a source situated at a great distance.

But before proceeding to consider a higher value of kc, it will be well to supplement the information already given under the head of $kc = 2$. The original calculation was limited to the principal values of μ, corresponding to the poles and the equator, under the impression that results for other values of μ would show nothing distinctive. The first suggestion to the contrary was from experiment. In observing the shadow of a sphere, by listening through a tube whose open end was presented to the sphere, it was found that the somewhat distant source was *more* loudly heard at the anti-pole ($\mu = -1$) than at points 40° or 50° therefrom. This is analogous to POISSON's experiment, where a bright point is seen in the centre of the shadow of a circular disc—an experiment easily imitated acoustically†—and it may be generally explained in the same manner. This led to further calculations for values of μ between 0 and -1, giving numbers in harmony with observation. The complete results for this case ($kc = 2$) are recorded in the annexed table. In obtaining them, terms of LEGENDRE's series, up to and including P_8, were retained. The angles θ are those whose cosine is μ.

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* See RAYLEIGH, 'Proc. Roy. Soc.,' vol. 72, p. 40; also MACDONALD, vol. 71, p. 251: vol. 72, p. 59; POINCARÉ, vol. 72, p. 42.

† 'Phil. Mag.,' vol. 9, p. 278, 1880; 'Scientific Papers,' vol. 1, p. 472.

$kc = 2.$

θ.	F + iG.	F² + G².	4 (F² + G²).
0°	$+\cdot 7968 + \cdot 2342\,i$	·6898	2·759
15	$+\cdot 8021 + \cdot 1775\,i$	·6749	2·700
30	$+\cdot 7922 + \cdot 0147\,i$	·6278	2·511
45	$+\cdot 7139 - \cdot 2287\,i$	·5619	2·248
60	$+\cdot 5114 - \cdot 4793\,i$	·4912	1·965
75	$+\cdot 1898 - \cdot 6247\,i$	·4263	1·705
90	$-\cdot 1538 - \cdot 5766\,i$	·3562	1·425
105	$-\cdot 3790 - \cdot 3413\,i$	·2601	1·040
120	$-\cdot 3992 - \cdot 0243\,i$	·1600	0·640
135	$-\cdot 2401 + \cdot 2489\,i$	·1196	0·478
150	$-\cdot 0088 + \cdot 4157\,i$	·1729	0·692
165	$+\cdot 1781 + \cdot 4883\,i$	·2701	1·080
180	$+\cdot 2495 + \cdot 5059\,i$	·3182	1·273

A plot of $4\,(F^2 + G^2)$ against θ is given in fig. 1, curve A.

The investigation for $kc = 10$ could probably be undertaken with success upon the lines explained in 'Theory of Sound;' but as it is necessary to include some 20 terms of the expansion in LEGENDRE's series, I considered that it would be advantageous to use certain formulæ of reduction by which the functions of various orders can be deduced from their predecessors, and this involves a change of notation. Formulæ convenient for the purpose have been set out by Professor LAMB.* The velocity-

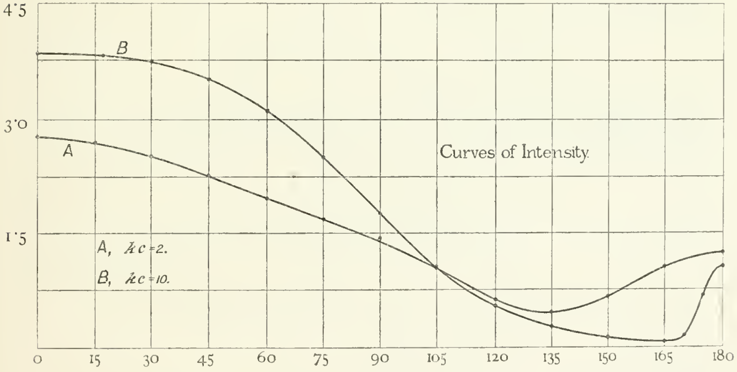

Fig. 1.

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* 'Hydrodynamics,' § 267 ; 'Camb. Phil. Trans.,' vol. 18, p. 350 1900.

potential ψ is supposed to be proportional throughout to e^{ikat}, but this time-factor is usually omitted. The general differential equation satisfied by ψ is

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2} + k^2\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

of which the solution in polar co-ordinates applicable to a *divergent* wave of the n^{th} order in LAPLACE's series may be written

$$\psi_n = S_n r^n \chi_n (kr) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

For the present purpose we may suppose without loss of generality that $k = 1$. The differential equation satisfied by $\chi_n (r)$ is

$$\frac{d^2\chi_n}{dr^2} + \frac{2n+2}{r}\frac{d\chi_n}{dr} + \chi_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and of this the solution which corresponds to a divergent wave is

$$\chi_n (r) = \left(-\frac{d}{r\,dr}\right)^n \frac{e^{-ir}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Putting $n = 0$ and $n = 1$, we have

$$\chi_0 (r) = \frac{e^{-ir}}{r}, \qquad \chi_1 (r) = \frac{(1 + ir)\,e^{-ir}}{r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

It is easy to verify that (4) satisfies (3). For if χ_n satisfies (3), $r^{-1}\chi'_n$ satisfies the corresponding equation for χ_{n+1}. And $r^{-1}e^{-ir}$ satisfies (3) when $n = 0$.

From (3) and (4) the following formulæ of reduction may be verified:

$$\chi'_n (r) = -\, r\chi_{n+1} (r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$r\chi'_n (r) + (2n + 1)\,\chi_n (r) = \chi_{n-1} (r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

$$\chi_{n+1} (r) = \frac{(2n + 1)\,\chi_n (r) - \chi_{n-1} (r)}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

By means of the last, χ_2, χ_3, &c., may be built up in succession from χ_0 and χ_1.

From (2)

$$d\psi_n/dr = S_n (nr^{n-1}\chi_n + r^n\chi'_n),$$

or, with use of (7),

$$d\psi_n/dr = r^{n-1}S_n \{\chi_{n-1} - (n + 1)\,\chi_n\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Thus, if U_n be the n^{th} component of the normal velocity at the surface of the sphere ($r = c$),

$$U_n = c^{n-1}S_n\{\chi_{n-1} (c) - (n + 1)\,\chi_n (c)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

When $n = 0$,

$$U_0 = S_0\chi'_0 (c) = -\, S_0 c\chi_1 (c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

The introduction of S_n from (10), (11) into (2) gives ψ_n in terms of U_n supposed known.

When r is very great in comparison with the wave-length, we get from (4)

$$\chi_n(r) = \frac{i^n e^{-ir}}{r^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

so that

$$\psi_n = S_n \frac{i^n e^{-ir}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

In order to find the effect at a great distance of a source of sound localised on the surface of the sphere at the point $\mu = 1$, we have only to expand the complete value of U in Legendre's functions. Thus

$$\begin{aligned} U_n &= \tfrac{1}{2}(2n+1) P_n(\mu) \int_{-1}^{+1} U P_n(\mu) \, d\mu \\ &= \tfrac{1}{2}(2n+1) P_n(\mu) \int_{-1}^{+1} U \, d\mu = \frac{2n+1}{4\pi c^2} P_n(\mu) \iint U \, dS \quad . \quad . \quad (14), \end{aligned}$$

in which $\iint U\,dS$ denotes the magnitude of the *source*, *i.e.*, the integrated value of U over the small area where it is sensible. The complete value of ψ may now be written

$$\psi = \frac{\iint U\,dS \, . \, e^{i(nt-r)}}{4\pi r} \sum \frac{(2n+1) i^n P_n(\mu)}{c^{n+1} \{\chi_{n-1}(c) - (n+1)\chi_n(c)\}} \quad . \quad . \quad . \quad (15).$$

When $n = 0$, $\chi_{n-1}(c) - (n+1)\chi_n(c)$ is to be replaced by $-c^2 \chi_1(c)$.

If we compare (15) with the corresponding expression in "Theory of Sound," (3), § 238, we get

$$c^{n+1} \{\chi_{n-1}(c) - (n+1)\chi_n(c)\} = -i^n e^{-ic} F_n(ic) \quad . \quad . \quad . \quad . \quad (16).$$

Another particular case of interest arises when the point of observation, as well as the source, is on the sphere, so that, instead of $r = \infty$, we have $r = c$. Equation (15) is then replaced by

$$\psi = \frac{\iint U\,dS \, . \, e^{ikat}}{4\pi c} \sum \frac{(2n+1)\chi_n(c) P_n(\mu)}{\chi_{n-1}(c) - (n+1)\chi_n(c)} \quad . \quad . \quad . \quad . \quad . \quad (17).$$

It may be remarked that, since ψ in (17) is infinite when $\mu = +1$ and accordingly $P_n = 1$, the convergence at other points can only be attained in virtue of the factors P_n. The difficulties in the way of a practical calculation from (17) may be expected to be greater than in the case of (15).

We will now proceed to the actual calculation for the case of $c = 10$, or $kc = 10$. The first step is the formation of the values of the various functions $\chi_n(10)$, starting from $\chi_0(10)$, $\chi_1(10)$. For these we have from (5)

$$10\chi_0(10) = \cos 10 - i \sin 10,$$

$$10^2\chi_1(10) = \tfrac{1}{10}\cos 10 + \sin 10 + i(\cos 10 - \tfrac{1}{10}\sin 10).$$

The angle (10 radians) $= 540° + 32° \, 57'{\cdot}468$; thus

$$\sin 10 = -\cdot 5440210, \quad \cos 10 = -\cdot 8390716,$$

and

$$10\chi_0 = -\cdot 8390716 + \cdot 5440210\, i,$$

$$10^2\chi_1 = -\cdot 6279282 - \cdot 7846695\, i.$$

From these, χ_2, χ_3, . . . are to be computed in succession from (8), which may be put into the form

$$10^{n+2}\chi_{n+1} = \frac{2n+1}{10}\, 10^{n+1}\chi_n - 10^n\chi_{n-1}.$$

For example,

$$10^3\chi_2 = \cdot 3\,(10^2\chi_1) - 10\chi_0 = +\cdot 6506931 - \cdot 7794218\, i.$$

When the various functions $10^{n+1}\chi_n$ have been computed, the next step is the computation of the denominators in (15). We write

$$\begin{aligned} \mathrm{D}_n &= 10^{n+1}\{\chi_{n-1} - (n+1)\chi_n\} \\ &= 10 \times 10^n\chi_{n-1} - (n+1)\,10^{n+1}\chi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (18), \end{aligned}$$

and the values of D_n are given along with $10^{n+1}\chi_n$ in the annexed table.

n.	$10^{n+1}\chi_n(10)$.	D_n.
0	$-\; 0{\cdot}83907 + 0{\cdot}54402\, i$	$+\; 6{\cdot}2793 + 7{\cdot}8467\, i$
1	$-\; 0{\cdot}62793 - 0{\cdot}78467\, i$	$-\; 7{\cdot}1349 + 7{\cdot}0095\, i$
2	$+\; 0{\cdot}65069 - 0{\cdot}77942\, i$	$-\; 8{\cdot}2314 - 5{\cdot}5084\, i$
3	$+\; 0{\cdot}95327 + 0{\cdot}39496\, i$	$+\; 2{\cdot}6938 - 9{\cdot}3741\, i$
4	$+\; 0{\cdot}01660 + 1{\cdot}05589\, i$	$+\; 9{\cdot}4498 - 1{\cdot}3299\, i$
5	$-\; 0{\cdot}93834 + 0{\cdot}55534\, i$	$+\; 5{\cdot}7960 + 7{\cdot}2269\, i$
6	$-\; 1{\cdot}04877 - 0{\cdot}44501\, i$	$-\; 2{\cdot}0420 + 8{\cdot}6685\, i$
7	$-\; 0{\cdot}42506 - 1{\cdot}13386\, i$	$-\; 7{\cdot}0872 + 4{\cdot}6208\, i$
8	$+\; 0{\cdot}41117 - 1{\cdot}25578\, i$	$-\; 7{\cdot}9512 - 0{\cdot}0366\, i$
9	$+\; 1{\cdot}12406 - 1{\cdot}00096\, i$	$-\; 7{\cdot}1288 - 2{\cdot}5482\, i$
10	$+\; 1{\cdot}72454 - 0{\cdot}64605\, i$	$-\; 7{\cdot}7293 - 2{\cdot}9031\, i$
11	$+\; 2{\cdot}49747 - 0{\cdot}35574\, i$	$-\; 12{\cdot}7243 - 2{\cdot}1916\, i$
12	$+\; 4{\cdot}01964 - 0{\cdot}17216\, i$	$-\; 27{\cdot}2807 - 1{\cdot}3194\, i$
13	$+\; 7{\cdot}55164 - 0{\cdot}07465\, i$	$-\; 65{\cdot}5265 - 0{\cdot}6764\, i$
14	$+\; 16{\cdot}36978 - 0{\cdot}02941\, i$	$-\; 170{\cdot}030 - 0{\cdot}3054\, i$
15	$+\; 39{\cdot}92071 - 0{\cdot}01062\, i$	$-\; 475{\cdot}033 - 0{\cdot}124\, i$
16	$+\; 107{\cdot}3844 - 0{\cdot}00353\, i$	$-\; 1426{\cdot}33 - 0{\cdot}047\, i$
17	$+\; 314{\cdot}45 - 0{\cdot}0010\, i$	$-\; 4586{\cdot}2 - 0{\cdot}017\, i$
18	$+\; 993{\cdot}19 - 0{\cdot}000\, i$	$-\; 15725{\cdot}0 - 0{\cdot}010\, i$
19	$+\; 3360{\cdot}3$	$-\; 57274$
20	$+\; 12112$	$-\; 220750$
21	$+\; 46299$	$-\; 897460$
22	$+\; 186974$	$-\; 38374 \times 10^2$

It will be seen that the imaginary part of $10^{n+1}\chi_n(10)$ tends to zero, as n increases. It is true that if we continue the calculation, having used throughout, say, 5 figures, we find that the terms begin to increase again. This, however, is but an imperfection of calculation, due to the increasing value of $\frac{1}{10}(2n+1)$ in the formula and consequent loss of accuracy, as each term is deduced from the preceding pair. Any doubt that may linger will be removed by reference to (4), according to which the imaginary term in question has the expression

$$-ir^{n+1}\left(-\frac{d}{r\,dr}\right)^n\frac{\sin r}{r}.$$

Now, if we expand $r^{-1}\sin r$ and perform the differentiations, the various terms disappear in order. For example, after the 25th operation we have

$$\left(-\frac{d}{r\,dr}\right)^{25}\frac{\sin r}{r}=\frac{50\,.\,48\,.\,.\,.\,4\,.\,2}{51\,!}-\frac{52\,.\,50\,.\,.\,.\,6\,.\,4}{53\,!}r^2+\frac{54\,.\,.\,.\,6}{55\,!}r^4-\&c.,$$

the first term being in every case positive and the subsequent terms alternately negative and positive. The series is convergent, since the numerical values of the terms continually diminish, the ratio of consecutive terms being (when $r=10$)

$$\frac{100}{2\,.\,53},\qquad\frac{100}{4\,.\,55},\qquad\frac{100}{6\,.\,57},\qquad\&c.$$

Accordingly the first term gives a limit to the sum of the series. On introduction of the factor 10^{n+1}, this becomes

$$\frac{10^{26}}{1\,.\,3\,.\,5\,.\,.\,.\,49\,.\,51},$$

i.e., approximately $10^{-8}\times 3{\cdot}0$. *A fortiori*, when n is greater than 25, the imaginary part of $10^{n+1}\chi_n(10)$ is wholly negligible.

We can now form the coefficients of P_n under the sign of summation in (15), *i.e.*, the values of

$$i^n(2n+1)D_n^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

For a reason that will presently appear, it is convenient to separate the odd and even values of n.

n.	$i^n(2n+1)D_n^{-1}$.	n.	$i^n(2n+1)D_n^{-1}$.
0	$+0 \cdot 06217 - 0 \cdot 07769\, i$	1	$+0 \cdot 21020 - 0 \cdot 21396\, i$
2	$+0 \cdot 41955 - 0 \cdot 28076\, i$	3	$+0 \cdot 68978 - 0 \cdot 19822\, i$
4	$+0 \cdot 93391 + 0 \cdot 13143\, i$	5	$+0 \cdot 92629 + 0 \cdot 74289\, i$
6	$+0 \cdot 33469 + 1 \cdot 42083\, i$	7	$-0 \cdot 96831 + 1 \cdot 48517\, i$
8	$-2 \cdot 13800 + 0 \cdot 00984\, i$	9	$-0 \cdot 84474 - 2 \cdot 36328\, i$
10	$+2 \cdot 38104 - 0 \cdot 89430\, i$	11	$+0 \cdot 30236 + 1 \cdot 75549\, i$
12	$-0 \cdot 91426 + 0 \cdot 04422\, i$	13	$-0 \cdot 00425 - 0 \cdot 41200\, i$
14	$+0 \cdot 17056 - 0 \cdot 00031\, i$	15	$+0 \cdot 00002 + 0 \cdot 06526\, i$
16	$-0 \cdot 02314 + 0 \cdot 00000\, i$	17	$-0 \cdot 00000 - 0 \cdot 00762\, i$
18	$+0 \cdot 00235$	19	$+0 \cdot 00068\, i$
20	$-0 \cdot 00018$	21	$-0 \cdot 00005\, i$
22	$+0 \cdot 00001$		

In the case of $\theta = 0$, or $\mu = +1$, the P's are all equal to $+1$, and we have nothing more to do than to add together all the terms in the above table. When $\theta = 180^\circ$, or $\mu = -1$, the even P's assume (as before) the value $+1$, but now the odd P's have a reversed sign and are equal to -1. If we add together separately the even and odd terms, and so obtain the two partial sums Σ_1 and Σ_2, then $\Sigma_1 + \Sigma_2$ will be the value of Σ for $\theta = 0$, and $\Sigma_1 - \Sigma_2$ will be the value of Σ for $\theta = 180$. And this simplification applies not merely to the special values 0 and 180, but to all intermediate pairs of angles. If $\Sigma_1 + \Sigma_2$ corresponds to θ, $\Sigma_1 - \Sigma_2$ will correspond to $180 - \theta$.

For 0 and 180 we find

$$\Sigma_1 = +1 \cdot 22870 + \cdot 35326\, i,$$

$$\Sigma_2 = +0 \cdot 31135 + \cdot 85436\, i;$$

whence for $\theta = 0$

$$\Sigma = 2(F + iG) = +1 \cdot 54005 + 1 \cdot 20762\, i,$$

and for $\theta = 180$

$$\Sigma = 2(F + iG) = +0 \cdot 91735 - 0 \cdot 50110\, i.$$

When $\theta = 90^\circ$, the odd P's vanish, and the even ones have the values

$$P_0 = 1, \qquad P_2 = -\tfrac{1}{2}, \qquad P_4 = \frac{1 \,.\, 3}{2 \,.\, 4}, \qquad P_6 = -\frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6}, \quad \&c.$$

For other values of θ we require tables of $P_n(\theta)$ up to about $n = 20$. That given by Professor PERRY* is limited to n less than 7, and the results are expressed only to 4 places of decimals. I have been fortunate enough to interest Professor A. LODGE in this subject, and the Appendix to this paper gives a table calculated by him containing the P's up to $n = 20$ inclusive, and for angles from 0° to 90° at intervals of 5°. As has already been suggested, the range from 0° to 90° practically covers that from 90° to 180°, inasmuch as

$$P_{2n}(90 + \theta) = P_{2n}(90 - \theta), \qquad P_{2n+1}(90 + \theta) = -P_{2n+1}(90 - \theta).$$

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* 'Phil. Mag.,' vol. 32, p. 516, 1891; see also FARR, vol. 49, p. 572, 1900.

In the table of coefficients it will be observed that the highest entry occurs at $n = 10$, in accordance with an anticipation expressed in a former paper.

As will readily be understood, the multiplication by P_n and the summations involve a good deal of arithmetical labour. These operations, as well as most of the preliminary ones, have been carried out in duplicate with the assistance of Mr. C. BOUTFLOWER, of Trinity College, Cambridge.

$kc = 10.$

θ.	$2(F + iG)$.	$4(F^2 + G^2)$.
0	$+1\cdot54005 + 1\cdot20762\,i$	3·8300
5	$+1\cdot58407 + 1\cdot14959\,i$	3·8309
10	$+1\cdot70186 + 0\cdot96603\,i$	3·8295
15	$+1\cdot84773 + 0\cdot63523\,i$	3·8176
30	$+1\cdot52622 - 1\cdot17708\,i$	3·7148
45	$-1\cdot13754 - 1\cdot48453\,i$	3·4978
60	$-0\cdot74695 + 1\cdot59745\,i$	3·1098
75	$+1\cdot45160 - 0\cdot62553\,i$	2·4984
90	$-1\cdot31954 - 0\cdot09924\,i$	1·75104
105	$+0\cdot94204 + 0\cdot41681\,i$	1·06117
120	$-0\cdot57769 - 0\cdot48417\,i$	0·56815
135	$+0\cdot29444 + 0\cdot43841\,i$	0·27890
150	$-0\cdot08146 - 0\cdot35600\,i$	0·13338
165	$-0\cdot12081 + 0\cdot28341\,i$	0·09492
170	$+0\cdot35454 + 0\cdot01457\,i$	0·12591
175	$+0\cdot76023 - 0\cdot34059\,i$	0·69395
180	$+0\cdot91735 - 0\cdot50110\,i$	1·09263

The results are recorded in the annexed table and in curve B, fig. 1. The intention had been to limit the calculations to intervals of 15°, but the rapid increase in $(F^2 + G^2)$ between 165° and 180° seemed to call for the interpolation of two additional angles. This increase, corresponding to the bright point in POISSON's experiment of the shadow of a circular *disc*, is probably the most interesting feature of the results. A plot is given in fig. 1, showing the relation between the angle θ, measured from the pole, and the *intensity*, proportional to $F^2 + G^2$. It should, perhaps, be emphasised that the effect here dealt with is the intensity of the *pressure* variation, to which some percipients of sound, *e.g.*, sensitive flames, are obtuse. Thus at the antipole a sensitive flame close to the surface would not respond to a distant source, since there is at that place no periodic *motion*, as is evident from the symmetry.

I now proceed to consider the case where the source, as well as the place of observation, are situated upon the sphere; but as this is more difficult than the preceding, I shall not attempt so complete a treatment. It will be supposed still that $kc = 10$.

The analytical solution is expressed in (17), which we may compare with (15). Restricting ourselves for the present to the factors under the sign of summation, we see that the coefficient of P_n in (17) is

$$\frac{(2n+1)\,c^{n+1}\chi_n(c)}{c^{n+1}\,\{\chi_{n-1}(c)-(n+1)\,\chi_n(c)\}}=\frac{(2n+1)\,c^{n+1}\chi_n}{\mathrm{D}_n},$$

while the corresponding coefficient in (15) is

$$\frac{(2n+1)\,i^n}{\mathrm{D}_n}.$$

If these coefficients be called C_n, C'_n respectively, we have

$$\mathrm{C}_n=i^{-n}c^{n+1}\chi_n(c)\,.\,\mathrm{C}'_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

in which the complex factors $c^{n+1}\chi_n(c)$, C'_n, for $c=10$, have already been tabulated. We find

n.	$\dfrac{(2n+1)\,10^{n+1}\chi_n}{\mathrm{D}_n}$.	n.	$\dfrac{(2n+1)\,10^{n+1}\chi_n}{\mathrm{D}_n}$.
0	$-0{\cdot}0099+0{\cdot}0990\,i$	1	$-0{\cdot}0306+0{\cdot}2999\,i$
2	$-0{\cdot}0542+0{\cdot}5097\,i$	3	$-0{\cdot}0835+0{\cdot}7358\,i$
4	$-0{\cdot}1233+0{\cdot}9883\,i$	5	$-0{\cdot}1827+1{\cdot}2817\,i$
6	$-0{\cdot}2813+1{\cdot}6390\,i$	7	$-0{\cdot}4666+2{\cdot}0956\,i$
8	$-0{\cdot}8667+2{\cdot}6889\,i$	9	$-1{\cdot}8111+3{\cdot}3152\,i$
10	$-3{\cdot}5284+3{\cdot}0805\,i$	11	$-4{\cdot}2766+1{\cdot}3796\,i$
12	$-3{\cdot}6673+0{\cdot}3351\,i$	13	$-3{\cdot}1110+0{\cdot}0629\,i$
14	$-2{\cdot}7920+0{\cdot}0100\,i$	15	$-2{\cdot}6051+0{\cdot}0014\,i$
16	$-2{\cdot}4844+0{\cdot}0001\,i$	17	$-2{\cdot}3998$
18	$-2{\cdot}3369$	19	$-2{\cdot}2881$
20	$-2{\cdot}2496$	21	$-2{\cdot}2183$
22	$-2{\cdot}1925$	23	$-2{\cdot}1711$
24	$-2{\cdot}1528$	25	$-2{\cdot}1374$
26	$-2{\cdot}1240$		

The product above tabulated shows marked signs of approaching the limit -2, as n increases; so that the series (17) is divergent when $\mathrm{P}_n=1$, *i.e.*, when $\theta=0$, as was of course to be expected. The interpretation may be followed further. By the definition of P_n, we have

$$\{1-2\alpha\,.\cos\theta+\alpha^2\}^{-\frac{1}{2}}=1+\mathrm{P}_1\,.\,\alpha+\mathrm{P}_2\,.\,\alpha^2+\ldots+\mathrm{P}_n\,.\,\alpha^n+\ldots \quad . \quad (21);$$

so that, if we put $\alpha=1$,

$$1+\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3+\ldots=\frac{1}{2\sin\left(\frac{1}{2}\theta\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Thus, when θ is small, and the series tends to be divergent, we get from (17)

$$\psi = -\frac{\iint \mathrm{U}\, d\mathrm{S} \,.\, e^{ikat}}{2\pi \,.\, 2c \sin\left(\frac{1}{2}\theta\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23);$$

and this is the correct value, seeing that $2c \sin(\frac{1}{2}\theta)$ represents the distance between the source and the point of observation, and that on account of the sphere the value of ψ is twice as great in the neighbourhood of the source as it would be were the source situated in the open.

When $\theta = 180°$, *i.e.*, at the point on the sphere immediately opposite to the source, the series converges, since P_n takes alternately the values $+1$ and -1. It will be convenient to re-tabulate continuously these values from $n = 18$ onwards without regard to sign and to exhibit the differences.

n.	Function.	First difference.	Second difference.	Third difference.
18	2·3369	—	—	—
19	2·2881	− ·0488	—	—
20	2·2496	− ·0385	+ ·0103	—
21	2·2183	− ·0313	+ ·0072	− ·0031
22	2·1925	− ·0258	+ ·0055	− ·0017
23	2·1711	− ·0214	+ ·0044	− ·0011
24	2·1528	− ·0183	+ ·0031	− ·0013
25	2·1374	− ·0154	+ ·0029	− ·0002
26	2·1240	− ·0134	+ ·0020	− ·0009

In summing the infinite series, we have to add together the terms as they actually occur up to a certain point and then estimate the value of the remainder. The simple addition is carried as far as $n = 21$ inclusive, and the result is for the even values of n

$$-18{\cdot}3939 + 9{\cdot}3506\, i,$$

and for the odd values

$$-19{\cdot}4734 + 9{\cdot}1721\, i,$$

or, with signs reversed to correspond with $\mathrm{P}_{2n+1}(180) = -1$,

$$+19{\cdot}4734 - 9{\cdot}1721\, i.$$

The complete sum up to $n = 21$ inclusive is thus

$$+1{\cdot}0795 + {\cdot}1785\, i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

The remainder is to be found by the methods of Finite Differences. The formula applicable to series of this kind may be written

$$\phi(0) - \phi(1) + \phi(2) - \ldots = \tfrac{1}{2}\phi(0) - \tfrac{1}{4}\Delta\phi(0) + \tfrac{1}{8}\Delta^2\phi(0) - \ldots,$$

in which we may put

$$\phi(0) = 2{\cdot}1925, \quad \phi(1) = 2{\cdot}1711, \quad \&c.$$

Thus

$$\phi(0) - \phi(1) + \ldots = +\ 1{\cdot}0962 + {\cdot}0054 + {\cdot}0004 = 1{\cdot}1020,$$

and for the actual remainder this is to be taken negatively. The sum of the infinite series for $\theta = 180°$ is accordingly

$$-\ {\cdot}0225 + {\cdot}1785\, i \quad \ldots \ldots \ldots \quad (25),$$

from which the intensity, represented by $({\cdot}0225)^2 + ({\cdot}1785)^2$, is proportional to ·03237. Referring to (17), we see that the amplitude of ψ is in this case

$$\frac{\iint \mathrm{U}\, d\mathrm{S}}{4\pi c} \times \sqrt{({\cdot}03237)} \quad \ldots \ldots \ldots \quad (26).$$

We may compare this with the amplitude of the vibration which would occur at the same place if the sphere were removed. Here

$$\frac{\iint \mathrm{U}\, d\mathrm{S}}{4\pi r} = \frac{\iint \mathrm{U}\, d\mathrm{S}}{4\pi c} \times \sqrt{({\cdot}25)} \quad \ldots \ldots \ldots \quad (27),$$

since $r = 2c$. The effect of the sphere is therefore to reduce the intensity in the ratio of ·25 to ·03237.

In like manner we may treat the case of $\theta = 90°$, *i.e.*, when the point of observation is on the equator. The odd P's now vanish and the even P's take signs alternately opposite. The following table gives the values required for the direct summation, *i.e.*, up to $n = 21$ inclusive :—

n.	$\dfrac{(2n+1)\,10^{n+1}\chi_n \,.\, \mathrm{P}_n(90)}{\mathrm{D}_n}$.	n.	$\dfrac{(2n+1)\,10^{n+1}\chi_n \,.\, \mathrm{P}_n(90)}{\mathrm{D}_n}$.
0	$-\ {\cdot}0099 + {\cdot}0990\, i$	2	$+\ {\cdot}0271 - {\cdot}2548\, i$
4	$-\ {\cdot}0462 + {\cdot}3706\, i$	6	$+\ {\cdot}0879 - {\cdot}5122\, i$
8	$-\ {\cdot}2370 + {\cdot}7353\, i$	10	$+\ {\cdot}8683 - {\cdot}7581\, i$
12	$-\ {\cdot}8273 + {\cdot}0756\, i$	14	$+\ {\cdot}5848 - {\cdot}0021\, i$
16	$-\ {\cdot}4879 + {\cdot}0000\, i$	18	$+\ {\cdot}4334$
20	$-\ {\cdot}3964$		
	$-2{\cdot}0047 + 1{\cdot}2805\, i$		$+2{\cdot}0015 - 1{\cdot}5272\, i$

The next three terms, written without regard to sign, and their differences are as follows:—

22	·3688	—	—
24	·3470	$-$ ·0218	—
26	·3292	$-$ ·0178	$+$ ·0040

The remainder is found, as before, to be

$$+\tfrac{1}{2}(\cdot 3688)+\tfrac{1}{4}(\cdot 0218)+\tfrac{1}{8}(\cdot 0040)=+\cdot 1903.$$

The sum of the infinite series from the beginning is accordingly

$$+\cdot 1871-\cdot 2467\,i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

in which

$$(\cdot 1871)^2+(\cdot 2467)^2=\cdot 09588.$$

The distance between the source and the point of observation is now $2c \sin 45^\circ = c\sqrt{2}$.

The intensity in the actual case is thus ·09588 as compared with ·5 if the sphere were away.

For other angular positions than those already discussed, not only would the arithmetical work be heavier on account of the factors P_n, but the remainder would demand a more elaborated treatment.

APPENDIX.

By Professor A. LODGE.

TABLE of Zonal Harmonics; *i.e.*, of the Coefficients of the Powers of x as far as P_{20} in the Expansion of $(1-2x\cos\theta+x^2)^{-\frac{1}{2}}$ in the form $1+P_1x+P_2x^2+\ldots+P_nx^n+\ldots$ for 5° Intervals in the Values of θ from 0° to 90°. The Table is calculated to 7 decimal places, and the last figure is approximate.

θ.	P_1 (= cos θ).	P_2.	P_3.	P_4.	P_5.
0°	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	·9961947	·9886059	·9772766	·9622718	·9436768
10	·9848078	·9547695	·9105688	·8532094	·7839902
15	·9659258	·8995191	·8041639	·6846954	·5471259
20	·9396926	·8245333	·6648847	·4749778	·2714918
25	·9063078	·7320907	·5016273	·2465322	+ ·0008795
30	·8660254	·6250000	·3247595	+ ·0234375	− ·2232722
35	·8191520	·5065151	+ ·1454201	− ·1714242	− ·3690967
40	·7660444	·3802362	− ·0252333	− ·3190044	− ·4196822
45	·7071068	·2500000	− ·1767767	− ·4062500	− ·3756505
50	·6427876	·1197638	− ·3002205	− ·4275344	− ·2544885
55	·5735764	− ·0065151	− ·3886125	− ·3851868	− ·0867913
60	·5000000	− ·1250000	− ·4375000	− ·2890625	+ ·0898437
65	·4226183	− ·2320907	− ·4452218	− ·1552100	·2381072
70	·3420201	− ·3245333	− ·4130083	− ·0038000	·3280672
75	·2588190	− ·3995191	− ·3448846	+ ·1434296	·3427278
80	·1736482	− ·4547695	− ·2473819	+ ·2659016	·2810175
85	·0871557	− ·4886059	− ·1290785	·3467670	·1576637
90	Nil	− ·5000000	Nil	·3750000	Nil

	P_6.	P_7.	P_8.	P_9.	P_{10}.
0	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	·9215975	·8961595	·8675072	·8358030	·8012263
10	·7044712	·6164362	·5218462	+ ·4227908	+ ·3214371
15	·3983060	+ ·2455411	+ ·0961844	− ·0427679	− ·1650562
20	+ ·0719036	− ·1072262	− ·2518395	− ·3516966	− ·4012692
25	− ·2039822	− ·3440850	− ·4062285	− ·3895753	− ·3052371
30	− ·3740235	− ·4101780	− ·3387755	− ·1895752	− ·0070382
35	− ·4114480	− ·3095600	− ·1154393	+ ·0965467	+ ·2541595
40	− ·3235708	− ·1006016	+ ·1386270	·2900130	·2973452
45	− ·1484376	+ ·1270581	·2983398	·2855358	+ ·1151123
50	+ ·0563782	·2854345	·2946824	+ ·1040702	− ·1381136
55	·2297230	·3190966	+ ·1421667	− ·1296151	− ·2692039
60	·3232421	·2231445	− ·0736389	− ·2678985	− ·1882286
65	·3138270	+ ·0422192	− ·2411439	− ·2300283	+ ·0323225
70	·2088770	− ·1485259	− ·2780153	− ·0475854	+ ·2192910
75	+ ·0431002	− ·2730500	− ·1702200	+ ·1594939	·2316302
80	− ·1321214	− ·2834799	+ ·0233080	·2596272	+ ·0646821
85	− ·2637801	− ·1778359	·2017462	·1912893	− ·1498947
90	− ·3125000	Nil	·2734376	Nil	− ·2460938

Table of Zonal Harmonics; *i.e.*, of the Coefficients of the Powers of x as far as P_{20} in the Expansion of $(1-2x\cos\theta+x^2)^{-\frac{1}{2}}$ in the form $1+P_1x+P_2x^2+\ldots+P_nx^n+\ldots$ for 5° Intervals in the Values of θ from 0° to 90°. The Table is calculated to 7 decimal places, and the last figure is approximate—continued.

θ.	P_{11}.	P_{12}.	P_{13}.	P_{14}.	P_{15}.
°					
0	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	·7639723	·7242508	·6822849	+ ·6383094	+ ·5925694
10	+ ·2199746	+ ·1205620	+ ·0252742	− ·0639478	− ·1453436
15	− ·2654901	− ·3402156	− ·3868998	− ·4048245	− ·3948856
20	− ·4001361	− ·3528461	− ·2682722	− ·1585374	− ·0376336
25	− ·1739692	− ·0223995	+ ·1215469	+ ·2332489	+ ·2952537
30	+ ·1607048	+ ·2732027	+ ·3066580	+ ·2584895	+ ·1465789
35	·3096940	+ ·2532528	+ ·1130760	− ·0565267	− ·1950586
40	+ ·1712040	− ·0211959	− ·1892595	− ·2599246	− ·2083112
45	− ·1041843	− ·2467193	− ·2393239	− ·0972709	+ ·0903925
50	− ·2640939	− ·1987621	− ·0019170	+ ·1821884	+ ·2281988
55	− ·1769491	+ ·0522404	+ ·2209602	+ ·1959135	+ ·0110216
60	+ ·0638713	·2337529	+ ·1658041	− ·0571737	− ·2100185
65	+ ·2351950	+ ·1608831	− ·0863490	− ·2197701	− ·0989734
70	+ ·1864450	− ·0787947	− ·2239288	− ·0745390	+ ·1597121
75	− ·0305439	− ·2274796	− ·0850288	+ ·1687887	+ ·1638193
80	− ·2145820	− ·1307104	+ ·1544264	+ ·1730902	− ·0860215
85	− ·1988401	+ ·1041876	+ ·2010073	− ·0629592	− ·1982155
90	Nil	+ ·2255858	Nil	− ·2094726	Nil
	P_{16}.	P_{17}.	P_{18}.	P_{19}.	P_{20}.
0	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	+ ·5453192	+ ·4968206	+ ·4473403	+ ·3971492	+ ·3465207
10	− ·2173739	− ·2787566	− ·3284945	− ·3658960	− ·3905880
15	− ·3594981	− ·3024136	− ·2284640	− ·1432466	− ·0527721
20	+ ·0801110	+ ·1815511	+ ·2560661	+ ·2965867	+ ·3002029
25	·2997862	+ ·2495290	+ ·1566049	+ ·0399984	− ·0780855
30	+ ·0036143	− ·1318805	− ·2254922	− ·2553464	− ·2169986
35	− ·2565851	− ·2244163	− ·1151189	+ ·0289684	+ ·1556356
40	− ·0654984	+ ·0986597	+ ·2088162	+ ·2180388	+ ·1273281
45	+ ·2150310	+ ·2100803	+ ·0857609	− ·0809310	− ·1930653
50	+ ·1133974	− ·0732822	− ·1986904	− ·1792842	− ·0359655
55	− ·1714205	− ·2012353	− ·0625381	+ ·1207910	+ ·1945128
60	− ·1498551	+ ·0522168	+ ·1922962	+ ·1377671	− ·0483584
65	+ ·1249926	+ ·1956924	+ ·0427633	− ·1501989	− ·1644051
70	+ ·1757158	− ·0336558	− ·1883363	− ·0935549	+ ·1165241
75	− ·0760903	− ·1924117	− ·0249700	+ ·1696995	+ ·1093683
80	− ·1912133	+ ·0165069	+ ·1861639	+ ·0473145	− ·1608344
85	+ ·0255526	+ ·1908789	+ ·0082151	− ·1794383	− ·0383005
90	+ ·1963808	Nil	− ·1854706	Nil	+ ·1761970

EXPLANATION OF THE METHODS OF COMPILING AND CHECKING THE ABOVE TABLES.

Calculation of the Even Orders.

The Zonal Harmonics of even order in the foregoing tables were calculated from the formula obtained by expanding $(1 - 2x \cos \theta + x^2)^{-\frac{1}{2}}$ by means of the form

$$(1 - xe^{i\theta})^{-\frac{1}{2}} (1 - xe^{-i\theta})^{-\frac{1}{2}},$$

i.e.,

$$(a_0 + a_1 xe^{i\theta} + \ldots + a_n x^n e^{ni\theta} + \ldots + a_{2n} x^{2n} e^{2ni\theta} + \ldots)$$
$$\times (a_0 + a_1 xe^{-i\theta} + \ldots + a_n x^n e^{-ni\theta} + \ldots + a_{2n} x^{2n} e^{-2ni\theta} + \ldots),$$

where

$$a_r = \frac{1 \,.\, 3 \,.\, 5 \ldots (2r - 1)}{2 \,.\, 4 \,.\, 6 \ldots 2r}, \quad \text{and} \quad a_0 = 1;$$

whence

$$\mathrm{P}_{2n}(\theta) = a_n^2 + 2a_{n-1}a_{n+1} \cos 2\theta + \ldots + 2a_0 a_{2n} \cos 2n\theta.$$

To calculate the coefficients $a_n^2, 2a_{n-1}a_{n+1}, \ldots 2a_0a_{2n}$, an auxiliary table of values of $\log_{10} a_r$ was formed from $r = 1$ to $r = 20$, to 8 decimal places; and a similar table of $\log_{10} 2a_r$ from $r = 0$ to $r = 9$; so as readily to combine them to form (to 7 decimal places) the logarithms of the required coefficients for different values of n.

The coefficients were then calculated to 7 decimal places from their logarithms, and checked for each value of n by seeing that they added up to unity in each case.

Next, a table of values of $\log \cos 2\theta$, $\log \cos 4\theta, \ldots$ to 7 decimals, was formed for all values of θ, at 5° intervals, from 5° to 90°. The addition of these to the logarithms of the corresponding coefficients gave the logarithms of the various terms (except as regards sign) in the above expansion of $\mathrm{P}_{2n}(\theta)$. From these logarithms the terms themselves were calculated to 7 decimals and tabulated, the positive terms in black, and the negative terms in red ink. The accuracy of these terms was checked by making use of the identities

$$\text{(1.)} \quad 2 \cos 60^\circ = 1,$$

$$\text{(2.)} \quad \cos 50^\circ + \cos 70^\circ = \cos 10^\circ,$$

$$\text{(3.)} \quad \cos 40^\circ + \cos 80^\circ = \cos 20^\circ.$$

This, in addition to the primary identity

$$a_n^2 + 2a_{n-1}a_{n+1} + \ldots + 2a_0a_{2n} = 1,$$

checked all the terms effectually except those which were multiples of cos 30°. These were checked by adding a number of them together and comparing their sum with the sum of the coefficients multiplied in a lump by cos 30°.

In these ways all the separate terms were ensured to be free from errors due to carelessness in taking proportional parts, or any other incidental errors.

Then the terms were added together for each value of θ in $P_{2n}(\theta)$ for a given value of n, so obtaining the values required for the actual table. By *adding* I mean to include also subtracting, the artifice of putting positive terms in black and negative terms in red being a great help in this part of the work. Errors in this work were corrected by adding all the values of $P_{2n}(\theta)$ from $\theta = 5^\circ$ to $\theta = 90^\circ$ for a given value of n, and comparing the result with the sum obtained in a different way (see note at the end of the second auxiliary table appended). Up to P_{12} the additions and subtractions and checkings were all done without mechanical aid, but for the later values of n, from P_{14} to P_{20}, I made use of an EDMONDSON's calculating machine which was very kindly lent to me by Professor McLEOD.

In this way all the even harmonics were calculated and were ensured to be free from errors, except those incidental to the last figure, which is, of course, only approximate, as the terms used in the calculation were evaluated to 7 decimals only. I am confident, however, that the last figure is never far from the real value, and that it would be more accurate in every case to retain it in numerical work with the tables than to omit it. The error is not usually more than ± 2 in the 7th place, and I am confident that it never exceeds ± 3, whereas omitting it would lead to a possibility of ± 5 in *addition* to its actual error, *i.e.*, to a maximum error of ± 8. I have assumed that there are very few numerical calculations requiring an accuracy greater than an approximate 7th decimal place, and that, therefore, the vastly increased difficulty which would have been caused by working throughout with 8 decimals would have been wasted labour.

Calculation of the Odd Orders, and Final Checking.

When the even orders were calculated, the question arose as to the best way of calculating the odd orders. P_1, of course, gave no difficulty, being merely $\cos\theta$. P_3, also, was quite easy to calculate directly from its value $\frac{1}{8}(3\cos\theta + 5\cos 3\theta)$, EDMONDSON's machine being used for the purpose. The remaining odd functions were calculated from the even ones by means of the identity

$$(2n-1)P_1P_{n-1} = nP_n + (n-1)P_{n-2}.$$

The accuracy of the results was checked by recalculating the even P's from the odd ones by means of the same formula. This clinched everything.

The mode of using this formula which I adopted, between $\theta = 5^\circ$ and $\theta = 60^\circ$ inclusive, was different from that adopted between $\theta = 65^\circ$ and $\theta = 85^\circ$ inclusive, so as to minimize the effect of 7th-figure inaccuracies as much as possible.

Up to $\theta = 60°$ I used it in the form

$$P_n = \frac{(n+1)\,P_{n+1} + nP_{n-1}}{(2n+1)\,P_1},$$

where P_1 varied from 1 to $\frac{1}{2}$; each P being thus dependent on its immediate predecessor and successor.

Beyond 60° I thought it better to use it in the progressive form

$$P_n = \frac{(2n-1)\,P_1 P_{n-1} - (n-1)\,P_{n-2}}{n},$$

each P being thus calculated from the two preceding orders.

I believe that in this way the maximum risk of a 7th-figure error occurs at 60°, when $P_1 = \frac{1}{2}$, and is not very great even there, whereas the exclusive use of either method would have greatly magnified the error at one end or other of the table.

Auxiliary Tables.

TABLE of Values of log a_r and log $2a_r$.

r.	log a_r.	log $2a_r$.
0	Nil	0·30103000
1	$\bar{1}\cdot69897000$	Nil
2	$\bar{1}\cdot57403127$	$\bar{1}\cdot87506126$
3	$\bar{1}\cdot49485002$	$\bar{1}\cdot79588002$
4	$\bar{1}\cdot43685807$	$\bar{1}\cdot73788807$
5	$\bar{1}\cdot39110058$	$\bar{1}\cdot69213058$
6	$\bar{1}\cdot35331202$	$\bar{1}\cdot65434202$
7	$\bar{1}\cdot32112734$	$\bar{1}\cdot62215734$
8	$\bar{1}\cdot29309862$	$\bar{1}\cdot59412861$
9	$\bar{1}\cdot26827503$	$\bar{1}\cdot56930503$
10	$\bar{1}\cdot24599864$	
11	$\bar{1}\cdot22579525$	
12	$\bar{1}\cdot20731185$	
13	$\bar{1}\cdot19027851$	
14	$\bar{1}\cdot17448424$	
15	$\bar{1}\cdot15976098$	
16	$\bar{1}\cdot14597270$	
17	$\bar{1}\cdot13300772$	
18	$\bar{1}\cdot12077326$	
19	$\bar{1}\cdot10919139$	
20	$\bar{1}\cdot09819601$	

TABLE showing the Acute Angles (in degrees) whose Cosines were required in Forming the Terms belonging to the Harmonics of Even Order. The Signs prefixed to the Angles Indicate whether their Cosines had to be Added or Subtracted.

θ.	2θ.	4θ.	6θ.	8θ.	10θ.	12θ.	14θ.	16θ.	18θ.	20θ.
5	+10	+20	+30	+40	+50	+60	+70	+80	±90	−80
10	+20	+40	+60	+80	−80	−60	−40	−20	− 0	−20
15	+30	+60	±90	−60	−30	− 0	−30	−60	∓90	+60
20	+40	+80	−60	−20	−20	−60	+80	+40	+ 0	+40
25	+50	−80	−30	−20	−70	+60	+10	+40	±90	−40
30	+60	−60	− 0	60	+60	+ 0	+60	−60	− 0	−60
35	+70	−40	−30	+80	+10	+60	−50	−20	∓90	+20
40	+80	−20	−60	+40	+40	−60	−20	+80	+ 0	+80
45	±90	− 0	∓90	+ 0	±90	− 0	∓90	+ 0	±90	− 0
50	−80	−20	+60	+40	−40	−60	+20	+80	− 0	+80
55	−70	−40	+30	+80	−10	+60	+50	−20	∓90	+20
60	−60	−60	+ 0	−60	−60	+ 0	−60	−60	+ 0	−60
65	−50	−80	+30	−20	+70	+60	−10	+40	±90	−40
70	−40	+80	+60	−20	+20	−60	−80	+40	− 0	+40
75	−30	+60	±90	−60	+30	− 0	+30	−60	∓90	+60
80	−20	+40	−60	+80	+80	−60	+40	−20	+ 0	−20
85	−10	+20	−30	+40	−50	+60	−70	+80	±90	−80
90	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0	+ 0

Note.—The terms in each of the columns headed 4θ, 8θ, 12θ, . . . all balance, their sum being zero.

The terms in each of the columns headed 2θ, 6θ, 10θ, . . . balance except the last term.

Hence the sum of all the 18 values of $P_n(\theta)$, from 5° to 90°,

$$= 18a_n^2 - 2a_{n-1}a_{n+1} - 2a_{n-3}a_{n+3} - \dots$$

This equivalence was made use of in finally checking the values of the harmonics of each even order.

Some Characteristics of the Functions as shown by the Tables.

The functions become more and more undulating as n increases, $P_n(\theta)$ having n zeroes between $\theta = 0°$ and 180°, similarly spaced on either side of 90°. The most remarkable peculiarity noticeable in drawing their graphs is that the intervals between the successive zeroes from the first to the n^{th} are almost exactly equal.

The graph of P_{20} is reproduced in fig. 2, to emphasize this peculiarity of equal intervals.

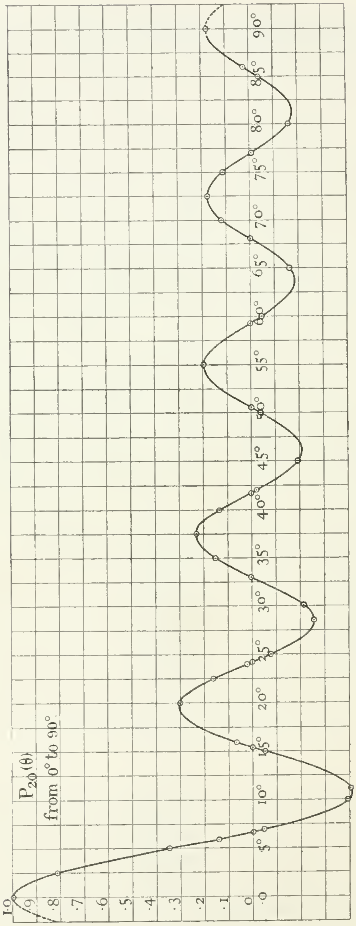
$P_{20}(\theta)$
from 0° to 90°
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
·0
5°
10°
15°
20°
25°
30°
35°
40°
45°
50°
55°
60°
65°
70°
75°
80°
85°
90°

Fig. 2.

In this respect LAPLACE's approximate formula for high values of n, viz. :—

$$P_n(\theta) = \frac{\sqrt{2}}{\sqrt{(n\pi \sin\theta)}} \cos\left(n\theta + \frac{\theta}{2} - \frac{\pi}{4}\right),$$

shows a wonderful resemblance to the actual functions even for quite low values of n.

The *numerical* values of this function are, indeed, not very near the true values even when $n = 20$, as will be seen by the following short table :—

	Approximation.	True value.
$P_{20}(15°)$	$-\cdot 04577$	$-\cdot 05277$
30°	$-\cdot 21851$	$-\cdot 21700$
45°	$-\cdot 19602$	$-\cdot 19307$
60°	$-\cdot 04962$	$-\cdot 04836$
75°	$+\cdot 11051$	$+\cdot 10937$
90°	$+\cdot 17841$	$+\cdot 17620$

But, though its numerical values are not very close, the positions of most of its zeroes are remarkably near the correct places. It can, of course, only be considered between 0° and 180°, since $\sin\theta$ becomes negative beyond these limits. But between these limits it has n zeroes, with $n - 1$ equal intervals between them, the first zero being at $\theta = 270° \div (2n + 1)$, and the interval between successive zeroes being $360° \div (2n + 1)$, the formula for the required values of θ being $\theta = (4r + 3)\,90° \div (2n + 1)$, for integer values of r from 0 to $n - 1$.

Taking $n = 20$, this would make the first zero approximately at 6° 35′, and the constant interval 8° 47′, very nearly; the roots given by the formula being, roughly,

$$6°\ 35',\ 15°\ 22',\ 24°\ 9',\ 32°\ 55\tfrac{1}{2}',\ 41°\ 42\tfrac{1}{2}',\ 50°\ 29',\ 59°\ 16',\ \ldots$$

The actual value of the first root of $P_{20}(\theta) = 0$ is slightly over 6° 43′, and intervals between successive roots are very nearly equal, varying between 8° 43′ and 8° 47′.

The first ten roots are, to something like the nearest minute, 6° 43′, 15° 26′, 24° 11′, 32° 57′, 41° 44′, 50° 30′, 59° 16′, 68° 3′, 76° 50′, and 85° 37′.

Professor PERRY has brought out a table of Zonal Harmonics to 4 decimals, for every degree, as far as P_7 ('Phil. Mag.,' December, 1891), and by help of this table I have calculated the first root, and the intervals between successive roots, for P_3 to P_7, to something like 1 minute accuracy. Their values, and the corresponding approximations obtained from LAPLACE's formula above, are given in the following table, showing how far they differ for these low values of n :—

	First root, and successive intervals.	Laplace's approximation.
	° ′	° ′
P_3	39 14 50 46 50 46	38 34 51 26
P_4	30 33 39 34½ 39 45 39 34½	30 0 40 0
P_5	25 1 32 24 32 35 32 35 32 24	24 33 32 49
P_6	21 11 27 26 27 35 27 36 27 35 27 26	20 46 27 41½
P_7	18 24 23 45 23 54 23 57 23 57 23 54 23 45	18 0 24 0

These examples indicate that the first root is always greater than the value given by the approximate formula, and the successive intervals are slightly less.

The actual roots of P_7 are, approximately,

$$18° 24', \quad 42° 9', \quad 66° 3', \quad 90°, \text{ \&c.},$$

and those given by the Laplace formula are

$$18°, \quad 42°, \quad 66°, \quad 90°,$$

so that the true roots are all a little ahead of those given by the Laplace formula, so long as θ is less than 90°. Beyond 90° the roots are, of course, similarly spaced in everse order.

NOTE BY LORD RAYLEIGH.

Professor LODGE's comparison of P_{20} with LAPLACE's approximate value suggests the question whether it is possible to effect an improvement in the approximate expression without entailing too great a complication. The following, on the lines of the investigation in TODHUNTER's 'Functions of LAPLACE, &c.,'* § 89, shows, I think, that this can be done.

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* MACMILLAN and Co., London, 1875.

We have

$$\mathrm{P}_n = \frac{4}{\pi k(2n+1)}\left\{\sin(n+1)\theta\right.$$

$$\left. + \frac{1.(n+1)}{1.(2n+3)}\sin(n+3)\theta + \frac{1.3.(n+1)(n+2)}{1.2.(2n+3)(2n+5)}\sin(n+5)\theta + \ldots\right\} \quad . \quad . \quad (\alpha),$$

with

$$\frac{1}{k} = \frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)} = \sqrt{(\pi n)}\,.\left\{1 + \frac{1}{8n} + \ldots\right\} \quad . \quad . \quad . \quad . \quad (\beta).$$

When n is great, approximate values may be used for the coefficients of the sines in (α). To obtain LAPLACE's expression it suffices to take

$$1, \quad \frac{1}{2}, \quad \frac{1.3}{2.4}, \quad \frac{1.3.5}{2.4.6}, \quad \&c.;$$

but now we require a closer approximation. Thus

$$\frac{1.(n+1)}{1.(2n+3)} = \frac{1}{2}\left(1 - \frac{1}{2n+2}\right),$$

$$\frac{1.3.(n+1).(n+2)}{1.2.(2n+3).(2n+5)} = \frac{1.3}{2.4}\left(1 - \frac{1}{2n+2} - \frac{1}{2n+4}\right),$$

and so on. If we write

$$x = 1 - \frac{1}{2n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma),$$

the coefficients are approximately

$$1, \quad \frac{1}{2}x, \quad \frac{1.3}{2.4}x^2, \quad \frac{1.3.5}{2.4.6}x^3, \quad \&c.,$$

and the series takes actually the form assumed by TODHUNTER for analytical convenience. In his notation

$$\mathrm{C} = t\cos\theta + \tfrac{1}{2}t^3\cos 3\theta + \frac{1.3}{2.4}t^5\cos 5\theta + \ldots,$$

$$\mathrm{S} = t\sin\theta + \tfrac{1}{2}t^3\sin 3\theta + \frac{1.3}{2.4}t^5\sin 5\theta + \ldots,$$

and

$$\mathrm{P}_n = \frac{4}{\pi k(2n+1)}\{\mathrm{C}\sin n\theta + \mathrm{S}\cos n\theta\},$$

where ultimately t is to be made equal to unity.

By summation of the series ($t < 1$),

$$\mathrm{C} = \frac{t}{\sqrt{\rho}}\cos(\theta + \tfrac{1}{2}\phi), \quad \mathrm{S} = \frac{t}{\sqrt{\rho}}\sin(\theta + \tfrac{1}{2}\phi),$$

where

$$\rho^2 = 1 - 2t^2 \cos 2\theta + t^4, \qquad \tan\phi = \frac{t^2 \sin 2\theta}{1 - t^2 \cos 2\theta} \quad . \quad . \quad . \quad . \quad (\delta).$$

For our purpose it is only necessary to write C/t and S/t for C and S respectively, and to identify t^2 with x in (γ). Thus

$$\mathrm{P}_n = \frac{4}{\pi k(2n+1)\sqrt{\rho}} \sin\left(n\theta + \theta + \tfrac{1}{2}\phi\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon),$$

ρ and ϕ being given by (δ). We find, with $t = 1 - \dfrac{1}{4n}$,

$$\rho^2 = 4\sin^2\theta\left(1 - \frac{1}{2n}\right),$$

so that

$$\sqrt{\rho} = \sqrt{(2\sin\theta)} \,.\left(1 - \frac{1}{8n}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\zeta);$$

and

$$\tan\phi = \frac{\sin 2\theta}{2\sin^2\theta + 1/2n},$$

whence

$$\phi = \frac{\pi}{2} - \theta - \frac{\cot\theta}{4n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\eta).$$

Using (ζ), (η), (β) in (ϵ) we get

$$\mathrm{P}_n = \frac{\sqrt{2}}{\sqrt{(\pi n \sin\theta)}}\left\{1 - \frac{1}{4n}\right\}.\cos\left\{n\theta + \frac{\theta}{2} - \frac{\pi}{4} - \frac{\cot\theta}{8n}\right\} \quad . \quad . \quad . \quad (\theta),$$

which is the expression required.

By this extension, not only is a closer approximation obtained, but the logic of the process is improved.

A comparison of values according to (θ) with the true values may be given in the case of n equal to 20.

VALUES of P_{20}.

θ.	True value.	According to (θ).
°		
15	− ·05277	− ·05320
30	− ·21700	− ·21712
45	− ·19307	− ·19306
60	− ·04836	− ·04834
75	+ ·10937	+ ·10937
90	+ ·17620	+ ·17618

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V. *On the Integrals of the Squares of Ellipsoidal Surface Harmonic Functions.*

By G. H. Darwin, *F.R.S., Plumian Professor and Fellow of Trinity College, in the University of Cambridge.*

Received December 2,—Read December 10, 1903.

This paper forms a sequel to three others published in Series A of the 'Philosophical Transactions,' namely, "On Ellipsoidal Harmonic Analysis," vol. 197, pp. 461–557, "On the Pear-shaped Figure of Equilibrium of a Rotating Mass of Liquid," vol. 198, pp. 301–331, and "On the Stability of the Pear-shaped Figure of Equilibrium, &c.," vol. 200, pp. 251–314. I shall refer to these three papers as "Harmonics," "The Pear-shaped Figure," and "Stability."

In "Harmonics," the functions being expressed approximately, approximate formulæ are found for the integrals over the surface of the ellipsoid of the squares of all the surface harmonics. These integrals are of course required whenever it is proposed to make practical use of this method of analysis, and the evaluation of them is therefore an absolutely essential step towards any applications.

The analysis used in the determination of some of these integrals was very complicated, and is probably susceptible of improvement. Such improvement might perhaps be obtained by the methods of the present paper, but I do not care to spend a great deal of time on an attempt merely to improve the analysis.

In "Harmonics" the symmetry which really subsists between the three factors of the solid harmonic functions was sacrificed with the object of obtaining convenient approximate forms, and I do not think it would have been possible to obtain such satisfactory results without this sacrifice.* But this course had the disadvantage of rendering it difficult to evaluate the integrals of the squares of the surface harmonics.

All the harmonic functions up to the third order inclusive are susceptible of rigorous algebraic expression; and indeed the same is true of some but not of all the functions of the fourth order. Accordingly in these cases rigorous expressions for the integrals should also be obtainable, and the object of the present paper is to complete the preceding investigation in this respect.

It will be well to begin by a restatement of the notation. That used in "Harmonics" was convenient for the approximate and asymmetrical expressions

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* See Appendix, below.

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involved, but the notation used in the two later papers seems preferable where the formulæ are rigorous and symmetrical.

In "Harmonics" the squares of the semi-axes of the ellipsoid were

$$a^2 = k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right),\ b^2 = k^2(\nu^2 - 1),\ c^2 = k^2\nu^2.$$

The rectangular coordinates were connected with ellipsoidal coordinates ν, μ, ϕ by

$$\frac{x^2}{k^2} = -\frac{1-\beta}{1+\beta}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)\left(\mu^2 - \frac{1+\beta}{1-\beta}\right)\cos^2\phi,$$

$$\frac{y^2}{k^2} = -(\nu^2 - 1)(\mu^2 - 1)\sin^2\phi,$$

$$\frac{z^2}{k^2} = \nu^2\mu^2\frac{1-\beta\cos 2\phi}{1+\beta}.$$

The three roots of the cubic

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$$

were

$$u_1 = k^2\nu^2,\ u_2 = k^2\mu^2,\ u_3 = k^2\frac{1-\beta\cos 2\phi}{1-\beta}.$$

Lastly ν ranges from ∞ to 0, μ between ± 1, ϕ from 0 to 2π.

In the two later papers I put

$$\kappa^2 = \frac{1-\beta}{1+\beta},\ \kappa'^2 = 1 - \kappa^2,\ \nu = \frac{1}{\kappa\sin\gamma},\ \mu = \sin\theta;$$

and for convenience I introduced an auxiliary constant β (easily distinguishable from the β of the previous notation) defined by $\sin\beta = \kappa\sin\gamma$.

The squares of the semi-axes of the ellipsoid were then

$$a^2 = \frac{k^2\cos^2\gamma}{\sin^2\beta},\ b^2 = \frac{k^2\cos^2\beta}{\sin^2\beta},\ c^2 = \frac{k^2}{\sin^2\beta}.$$

The rectangular coordinates became

$$\frac{x^2}{k^2} = \frac{\cos^2\gamma}{\sin^2\beta}(1-\kappa^2\sin^2\theta)\cos^2\phi,\ \frac{y^2}{k^2} = \frac{\cos^2\beta}{\sin^2\beta}\cos^2\theta\sin^2\phi,\ \frac{z^2}{k^2} = \frac{1}{\sin^2\beta}\sin^2\theta\,(1-\kappa'^2\cos^2\phi).$$

The roots of the cubic were

$$u_1 = \frac{k^2}{\sin^2\beta},\ u_2 = k^2\sin^2\theta,\ u_3 = \frac{k^2}{\kappa^2}(1-\kappa'^2\cos^2\phi).$$

This is the notation which will be used in the present paper.

If $d\sigma$ be an element of surface of the ellipsoid, and p the central perpendicular on to the tangent plane, it appears from the formula at the foot of p. 257 of "Stability" that

$$\frac{p\,d\sigma}{d\theta\,d\phi} = \frac{k^3 \cos\beta \cos\gamma}{\sin^3\beta} \cdot \frac{\kappa^2 \cos^2\theta + \kappa'^2 \sin^2\phi}{\Delta\Gamma},$$

where $\Delta^2 = 1 - \kappa^2 \sin^2\theta$, $\Gamma^2 = 1 - \kappa'^2 \cos^2\phi$.

In the previous papers I have expressed the two factors of which a surface harmonic consists by $\mathfrak{P}_i^s(\mu)$ or $\mathsf{P}_i^s(\mu)$, and $\mathfrak{C}_i^s(\phi)$, $\mathsf{C}_i^s(\phi)$, $\mathfrak{S}_i^s(\phi)$ or $\mathsf{S}_i^s(\phi)$, one of the two P-functions being multiplied by one of the four cosine or sine functions.

Taking a pair of typical cases, the integrals to be evaluated are

$$\int (\mathfrak{P}_i^s\,\mathfrak{C}_i^s)^2 p\,d\sigma \qquad \text{and} \qquad \int (\mathfrak{P}_i^s\,\mathfrak{S}_i^s)^2 p\,d\sigma.$$

As it will be convenient to use an abridged notation, I will write these integrals $I_i^s(\cos)$ and $I_i^s(\sin)$, according to an easily intelligible notation.

These functions involve integrals of even functions, and therefore we may integrate through one octant of space, the limits of θ and ϕ being $\frac{1}{2}\pi$ to 0, and multiply the result by 8.

It is clear then that

$$I_i^s(\cos) = \frac{8k^3 \cos\beta \cos\gamma}{\sin^3\beta} \left[\int_0^{\frac{1}{2}\pi} \frac{\kappa^2 \cos^2\theta\,(\mathfrak{P}_i^s)^2}{\Delta}\,d\theta \int_0^{\frac{1}{2}\pi} \frac{(\mathfrak{C}_i^s)^2}{\Gamma}\,d\phi \right.$$
$$\left. + \int_0^{\frac{1}{2}\pi} \frac{(\mathfrak{P}_i^s)^2}{\Delta}\,d\theta \int_0^{\frac{1}{2}\pi} \frac{\kappa'^2 \sin^2\phi\,(\mathfrak{C}_i^s)^2}{\Gamma}\,d\phi \right].$$

Similar expressions are applicable to all the other forms of function, but we may proceed with this form as a type of all the others.

This formula shows that the variables are separable, and since we might substitute $\frac{1}{2}\pi - \psi$ for ϕ without changing the result, the ϕ integrals are of the same type as the θ integrals.

It has been stated above that two of the roots of the cubic equation are proportional to $\kappa^2 \sin^2\theta$ and $(1 - \kappa'^2 \cos^2\phi)$. By the nature of the harmonic functions it follows that if $[\mathfrak{P}_i^s(\mu)]^2$ is proportional to a certain function of $\kappa^2 \sin^2\theta$, $[\mathfrak{C}_i^s(\phi)]^2$ is proportional to the same function of $(1 - \kappa'^2 \cos^2\phi)$.

It follows that if $(\mathfrak{P}_i^s)^2 = F(\kappa^2 - \kappa^2 \sin^2\theta) = F(\kappa^2 \cos^2\theta)$,

$$(\mathfrak{C}_i^s)^2 = \alpha F(\kappa^2 - 1 + \kappa'^2 \cos^2\phi) = \alpha F(-\kappa'^2 \sin^2\phi),$$

where α is a constant, which for the present we may regard as being unity. If then

$$[\mathfrak{P}_i^s(\mu)]^2 = A_0 + A_1\kappa^2 \cos^2\theta + A_2\kappa^4 \cos^4\theta + A_3\kappa^6 \cos^6\theta + \ldots,$$

we must have

$$[\mathfrak{C}_i^s(\phi)]^2 = A_0 - A_1\kappa'^2 \sin^2\phi + A_2\kappa'^4 \sin^4\phi - A_3\kappa'^6 \sin^6\phi + \ldots$$

Accordingly if there is a term $A_n \int \kappa^{2n+2} \cos^{2n+2} \theta \frac{d\theta}{\Delta}$ in $\int \kappa^2 \cos^2 \theta \, (\mathfrak{P}_i^s)^2 \frac{d\theta}{\Delta}$ and a term $(-)^m A_m \int \kappa'^{2m} \sin^{2m} \phi \frac{d\phi}{\Gamma}$ in $\int (\mathfrak{C}_i^s)^2 \frac{d\phi}{\Gamma}$, then there must be a term $A_m \int \kappa^{2m} \cos^{2m} \theta \frac{d\theta}{\Delta}$ in $\int (\mathfrak{P}_i^s)^2 \frac{d\theta}{\Delta}$, and a term $(-)^n A_n \int \kappa'^{2n+2} \sin^{2n+2} \phi \frac{d\phi}{\Gamma}$. It follows that the coefficient of $(-)^m A_n A_m$ in $I_i^s (\cos) \div \frac{8k^3 \cos\beta \cos\gamma}{\sin^3 \beta}$ is

$$\int \kappa^{2n+2} \cos^{2n+2} \theta \frac{d\theta}{\Delta} \int \kappa'^{2m} \sin^{2m} \phi \frac{d\phi}{\Gamma} - (-)^{n-m+1} \int \kappa^{2m} \cos^{2m} \theta \frac{d\theta}{\Delta} \int \kappa'^{2n+2} \sin^{2n+2} \phi \frac{d\phi}{\Gamma}.$$

For the sake of brevity I call this function $[2n+2,\ 2m]$, and we may state that one term in the required expression is $(-)^m A_n A_m [2n+2,\ 2m]$, where $[\ \ ]$ indicates the above function of the four integrals. It follows that $I_i^s (\cos) \div \frac{8k^3 \cos\beta \cos\gamma}{\sin^3 \beta} =$

$$\left.\begin{array}{r} A_0^2 [2, 0] - A_0 A_1 [2, 2] + A_0 A_2 [2, 4] - \ldots \\ + A_1 A_0 [4, 0] - A_1^2 [4, 2] + A_1 A_2 [4, 4] - \ldots \\ + A_2 A_0 [6, 0] - A_2 A_1 [6, 2] + A_2^2 [6, 4] - \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad . \quad . \quad . \quad . \quad (1).$$

Since

$$[2n, 2m] = \int \kappa^{2n} \cos^{2n} \theta \frac{d\theta}{\Delta} \int \kappa'^{2m} \sin^{2m} \phi \frac{d\phi}{\Gamma} - (-)^{n-m} \int \kappa^{2m} \sin^{2m} \theta \frac{d\theta}{\Delta} \int \kappa'^{2n} \sin^{2n} \phi \frac{d\phi}{\Gamma},$$

it is clear that

$$[2n, 2m] = -(-)^{n-m} [2m, 2n].$$

Hence if n and m differ by an odd number $[2n, 2m] = [2m, 2n]$, and if they differ by an even number $[2n, 2m] = -[2m, 2n]$. Also $[2n, 2n] = 0$.

Let us write $\{2n\} = \int_0^{\frac{1}{2}\pi} \kappa^{2n} \cos^{2n} \theta \frac{d\theta}{\Delta}$, $\{2n\}' = \int_0^{\frac{1}{2}\pi} \kappa'^{2n} \sin^{2n} \phi \frac{d\phi}{\Gamma}$, so that

$$[2n, 2m] = \{2n\} \{2m\}' - (-)^{n-m} \{2n\}' \{2m\}.$$

We must now evaluate these functions.

Since $\Delta^2 = 1 - \kappa^2 \sin^2 \theta$, we have by differentiation

$$\frac{d}{d\theta} [\Delta \sin\theta \cos^{2n-3} \theta] =$$

$$\frac{1}{\Delta} \{(2n-1)\kappa^2 \cos^{2n} \theta - (2n-2)(\kappa^2 - \kappa'^2) \cos^{2n-2} \theta - (2n-3) \kappa'^2 \cos^{2n-4} \theta\}.$$

Integrating between $\frac{1}{2}\pi$ and 0 and multiplying by κ^{2n-2} we have

$$\{2n\} = \frac{2n-2}{2n-1} (\kappa^2 - \kappa'^2) \{2n-2\} + \frac{2n-3}{2n-1} \kappa^2 \kappa'^2 \{2n-4\},$$

and by symmetry

$$\{2n\}' = -\frac{2n-2}{2n-1}(\kappa^2-\kappa'^2)\{2n-2\}' + \frac{2n-3}{2n-1}\kappa^2\kappa'^2\{2n-4\}'.$$

Multiplying the first of these by $\{2m\}'$ and the second by $-(-)^{n-m}\{2m\}$ and adding together we have

$$[2n,\,2m] = \frac{2n-2}{2n-1}(\kappa^2-\kappa'^2)[2n-2,\,2m] + \frac{2n-3}{2n-1}\kappa^2\kappa'^2[2n-4,\,2m].$$

By successive applications of this formula we may reduce any function $[2n,\,2m]$ until it depends on $[2,\,0]$, but the result becomes very complicated after a few successive reductions.

Now $\{0\} = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{\Delta} = \mathrm{F}, \quad \{0\}' = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{\Gamma} = \mathrm{F}'$

$$\{2\} = \int_0^{\frac{1}{2}\pi}\frac{\kappa^2\cos^2\theta}{\Delta}\,d\theta = \mathrm{E}-\mathrm{F}\kappa'^2, \quad \{2\}' = \int_0^{\frac{1}{2}\pi}\frac{\kappa'^2\sin^2\phi}{\Gamma}\,d\phi = \mathrm{E}'-\mathrm{F}'\kappa^2.$$

Then

$$\begin{aligned}[2,\,0] &= \{2\}\{0\}' - (-)^1\{2\}'\{0\} \\ &= \mathrm{EF}' + \mathrm{E'F} - \mathrm{FF}'.\end{aligned}$$

But it is well known that this combination of the complete elliptic integrals with moduli κ and κ' is $\frac{1}{2}\pi$.*

Hence $[2,\,0] = \frac{1}{2}\pi$.

It seems unnecessary to reproduce the simple algebra involved in the successive reductions, and I therefore merely give the results, as follows:—

$$[0,2]=[2,0]=\tfrac{1}{2}\pi \qquad [2,4]=[4,2]=\tfrac{1}{3}\kappa^2\kappa'^2.\tfrac{1}{2}\pi$$

$$-[0,4]=[4,0]=\tfrac{2}{3}(\kappa^2-\kappa'^2).\tfrac{1}{2}\pi \qquad -[2,6]=[6,2]=\tfrac{4}{3.5}(\kappa^2-\kappa'^2)\kappa^2\kappa'^2.\tfrac{1}{2}\pi$$

$$[0,6]=[6,0]=\tfrac{1}{3.5}(8-23\kappa^2\kappa'^2).\tfrac{1}{2}\pi \qquad [2,8]=[8,2]=\tfrac{1}{3.5.7}(24-71\kappa^2\kappa'^2)\kappa^2\kappa'^2.\tfrac{1}{2}\pi$$

$$-[0,8]=[8,0]=\tfrac{8}{3.5.7}(\kappa^2-\kappa'^2)(6-11\kappa^2\kappa'^2).\tfrac{1}{2}\pi$$

$$[4,6]=[6,4]=\tfrac{1}{5}\kappa^4\kappa'^4.\tfrac{1}{2}\pi \qquad [6,8]=[8,6]=\tfrac{1}{7}\kappa^6\kappa'^6.\tfrac{1}{2}\pi$$

$$-[4,8]=[8,4]=\tfrac{6}{5.7}(\kappa^2-\kappa'^2)\kappa^4\kappa'^4.\tfrac{1}{2}\pi$$

These are the only functions of this kind which are needed for the evaluations of integrals in this paper.

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[*] See for example DURÈGE'S 'Theorie der Elliptischen Functionen,' p. 293.

When these functions are introduced into (1) and the terms re-arranged, I find :—

$$
\begin{aligned}
I_i^s(\cos) \div \frac{4\pi k^3 \cos\beta\cos\gamma}{\sin^3\beta} = {}& \\
& A_0^2 - \tfrac{1}{3}\kappa^2\kappa'^2 A_1^2 + \tfrac{1}{5}\kappa^4\kappa'^4 A_2^2 - \tfrac{1}{7}\kappa^6\kappa'^6 A_3^2 + \tfrac{1}{9}\kappa^8\kappa'^8 A_4^2 - \dots \\
& + 2(\kappa^2 - \kappa'^2)\left[\tfrac{1}{1.3} A_0A_1 - \tfrac{2}{3.5}\kappa^2\kappa'^2 A_1A_2 + \tfrac{3}{5.7}\kappa^4\kappa'^4 A_2A_3 - \tfrac{1}{7.9}\kappa^6\kappa'^6 A_3A_4 + \dots\right] \\
& + \tfrac{2}{1.3.5}\left[\tfrac{1}{2}(2.4) - 3^2\kappa^2\kappa'^2\right] A_0A_2 - \tfrac{2}{3.5.7}\left[\tfrac{1}{2}(4.6) - 5^2\kappa^2\kappa'^2\right]\kappa^2\kappa'^2 A_1A_3 \\
& + \tfrac{2}{5.7.9}\left[\tfrac{1}{2}(6.8) - 7^2\kappa^2\kappa'^2\right]\kappa^4\kappa'^4 A_2A_4 - \tfrac{2}{7.9.11}\left[\tfrac{1}{2}(8.10) - 9^2\kappa^2\kappa'^2\right]\kappa^6\kappa'^6 A_3A_5 + \dots \\
& + 2(\kappa^2 - \kappa'^2)\left[\tfrac{2}{1.3.5.7}(3.5(1-\kappa^2\kappa'^2) - 3) A_0A_3 - \tfrac{3}{3.5.7.9}(5.7(1-\kappa^2\kappa'^2) - 3)\kappa^2\kappa'^2 A_1A_4\right. \\
& \qquad\qquad \left. + \tfrac{4}{5.7.9.11}(7.9(1-\kappa^2\kappa'^2) - 3)\kappa^4\kappa'^4 A_2A_5 - \dots\right] \\
& + \tfrac{2}{1.3.5.7.9}(192 - 816\kappa^2\kappa'^2 + 525\kappa^4\kappa'^4) A_0A_4 - \dots \qquad (2).
\end{aligned}
$$

In this result a good many terms are added which are not deducible from the table of functions given above, but every term as stated here has actually been computed. The laws governing the succession of terms in the first six lines seem clear, but I do not claim that the proof of the laws is rigorous. I do not perceive how each series is derived from those preceding it, and I have no idea how the series beginning with A_0A_4 would go on. With sufficient patience it would no doubt be possible to determine the general law of the series, but I do not propose to make the attempt at present, since we have more than enough for the immediate object in view.

This result (2) is, of course, equally applicable to the integrals of the type I_i^s (sin).

In order to effect the required integrations we must define the functions, and I take the definitions (with a few very slight changes) from § 2 of "the Pear-shaped Figure." In order to use the preceding analysis it is necessary that the square of the P-function and the square of the cosine or sine function should be the same functions of $\kappa^2\cos^2\theta$ and of $-\kappa'^2\sin^2\phi$. But as in the definitions to be used this symmetry does not hold good, a difficulty arises, which may, however, be easily overcome. If the P-function be multiplied by any factor f, and the cosine or sine function by any factor g, the integral will be multiplied by f^2g^2. I therefore introduce such factors f and g as will render the residual factors of the squares of the P and cosine or sine-functions symmetrical in the proper manner.

It seems desirable to show how the results found here accord with the approximate integrals as found on pp. 548-9, § 22, of "Harmonics." In this connection I remark that $\dfrac{k^3\cos\beta\cos\gamma}{\sin^3\beta}$, when written in the notation of "Harmonics," is $k^3\nu(\nu^2-1)^{\frac{1}{2}}\left(\nu^2 - \dfrac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, a factor which I denoted in that paper by M.

It does not seem necessary to give full details of the analysis in the several cases, since it is sometimes tedious, and it merely involves the substitution in the formula of the values of A_0, A_1, A_2, &c.

We will now take the several harmonics successively.

HARMONIC OF THE ORDER ZERO.

This harmonic is simply unity, so that $A_0 = 1$ and all other A's vanish. The formula is

$$I_0\,(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{\sin^3\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This is obviously right since the integral is $\int p\,d\sigma$, of which this is the known value.

HARMONICS OF THE FIRST ORDER.

Here we have all the A's zero excepting A_0 and A_1, and when the functions have the proper symmetrical forms, we have from (2),

$$I_i^s\binom{\cos}{\sin} = \frac{4\pi k^3 \cos\beta \cos\gamma}{\sin^3\beta}\left[A_0^2 - \tfrac{1}{3}\,\kappa^2\kappa'^2 A_1^2 + \tfrac{2}{3}\,(\kappa^2 - \kappa'^2)\,A_0 A_1\right],\ (s = 0,\ 1).$$

(1) *The Zonal Harmonic.*

I define this thus,

$$\left.\begin{aligned} \mathfrak{P}_1\,(\mu) &= \sin\theta = f\,.\,(\kappa^2 - \kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathsf{C}_1\,(\phi) &= (1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} = g\,.\,(\kappa^2 + \kappa'^2\sin^2\phi)^{\frac{1}{2}}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

where $f = \dfrac{1}{\kappa}$, $g = 1$.

On squaring $\mathfrak{P}_1\,(\mu)$, it is clear that

$$A_0 = \kappa^2, \qquad A_1 = -1.$$

Whence I find

$$I_1\,(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta}\,.\,f^2 g^2 \kappa^2.$$

Since with definition (4) $f^2 g^2 \kappa^2 = 1$,

$$I_1\,(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

In "Harmonics" this harmonic is defined by

$$\mathfrak{P}_1\,(\mu) = P_1\,(\mu) = \mu\,; \qquad \mathsf{C}_1\,(\phi) = \sqrt{(1 - \beta\cos 2\phi)} \quad . \quad . \quad . \quad . \quad (6).$$

Now we must take for f and g values such as to bring the two definitions into accord. This is the case if

$$f = \frac{1}{\kappa} = \sqrt{\frac{1+\beta}{1-\beta}}, \qquad g = \sqrt{1+\beta};$$

and $f^2 g^2 \kappa^2 = 1 + \beta$.

Hence

$$I_1(\cos) = \frac{4\pi\mathrm{M}}{3}(1+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

agreeing with the result on p. 549 of "Harmonics" for the case $i = 1$, $s = 0$, type OEC.

(2) *The Sectorial Cosine Harmonic.*

I define this thus,

$$\left.\begin{aligned} \mathsf{P}_1^1(\mu) &= (1 - \kappa^2 \sin^2\theta)^{\frac{1}{2}} = f\,.\,(\kappa'^2 + \kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathfrak{C}_1^1(\phi) &= \cos\phi \qquad\qquad\quad\;\; = g\,.\,(\kappa'^2 - \kappa'^2 \sin^2\phi)^{\frac{1}{2}}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8),$$

where $f = 1$, $g = \dfrac{1}{\kappa'}$.

By symmetry with the last result

$$I_1^1(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta}\,.\,f^2 g^2 \kappa'^2 = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta} \quad . \quad . \quad . \quad . \quad (9).$$

In "Harmonics" I defined the functions thus,

$$\left.\begin{aligned} \mathsf{P}_1^1(\mu) &= P_1^1(\mu)\left(\frac{\frac{1+\beta}{1-\beta} - \mu^2}{1-\mu^2}\right)^{\frac{1}{2}} = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(1 - \frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}}, \\ \mathfrak{C}_1^1(\phi) &= \cos\phi \end{aligned}\right\} \quad . \quad (10).$$

If we take $f = \left(\dfrac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, $g\kappa' = 1$, the two definitions agree, and we have

$$I_1^1(\cos) = \tfrac{4}{3}\pi\mathrm{M}\,\frac{1+\beta}{1-\beta} = \tfrac{4}{3}\pi\mathrm{M}\,(1 + 2\beta + 2\beta^2) \quad . \quad . \quad . \quad . \quad (11).$$

This agrees with the result on p. 549 of "Harmonics" with $i = 1$, $s = 1$, type OOC.

(3) *The Sectorial Sine Harmonic.*

I define this thus,

$$\left.\begin{aligned} \mathfrak{P}_1^1(\mu) &= \cos\theta = f\,.\,\kappa\cos\theta, \\ \mathfrak{S}_1^1(\phi) &= \sin\phi = g\,.\,\sqrt{-1}\,.\,\kappa'\sin\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

where $f = \dfrac{1}{\kappa}$, $g = \dfrac{1}{\kappa'\sqrt{-1}}$.

On squaring $\mathfrak{P}_1^1$ we find $A_0 = 0$, $A_1 = 1$, and

$$I_1^1(\sin) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta}(-f^2 g^2 \kappa^2 \kappa'^2) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta} \quad . \quad . \quad (13)$$

In "Harmonics" the definitions were the same, and therefore

$$I_1^1(\sin) = \frac{4\pi M}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

This agrees with the result on p. 548 of "Harmonics" with $i = 1$, $s = 1$, type OOS.

Harmonics of the Second Order.

In these the only coefficients are A_0, A_1, A_2, and (2) becomes.

$$I_2^s\binom{\cos}{\sin} = \frac{4\pi k^3\cos\beta\cos\gamma}{\sin^3\beta}\left[A_0^2 - \tfrac{1}{3}\kappa^2\kappa'^2 A_1^2 + \tfrac{1}{5}\kappa^4\kappa'^4 A_2^2\right.$$
$$\left. + \tfrac{2}{3}(\kappa^2 - \kappa'^2) A_0 A_1 - \tfrac{4}{15}(\kappa^2 - \kappa'^2)\kappa^2\kappa'^2 A_1 A_2 + \tfrac{2}{15}(4 - 9\kappa^2\kappa'^2) A_0 A_2\right],$$

with $s = 0, 1, 2$.

(1) *and* (4) *The Zonal and Sectorial Cosine Harmonics.*

These are defined thus,

$$\left.\begin{aligned} \mathfrak{P}_2^s(\mu) &= \kappa^2\sin^2\theta - q^2, \\ \mathfrak{C}_2^s(\phi) &= q'^2 - \kappa'^2\cos^2\phi, \quad (s = 0, 2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where $q^2 = \frac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$, with upper sign for $s = 0$ and lower for $s = 2$; and $q'^2 = 1 - q^2$.

Writing

$$t^2 = \kappa^2 - q^2 = q'^2 - \kappa'^2 = \tfrac{1}{3}[\kappa^2 - \kappa'^2 \pm (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}],$$

$$\left.\begin{aligned} \mathfrak{P}_2^s(\mu) &= f.(t^2 - \kappa^2\cos^2\theta), \\ \mathfrak{C}_2^s(\phi) &= g.(t^2 + \kappa'^2\sin^2\phi), \quad (s = 0, 2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where $f = 1$, $g = 1$. It may be noted that t^2 is a symmetrical function in κ^2 and $-\kappa'^2$.

Squaring $\mathfrak{P}_2^s$ we find

$$A_0 = t^4, \quad A_1 = -2t^2, \quad A_2 = 1.$$

After reduction I find, for $s = 0, 2$,

$$I_2^s(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{3 \sin^3\beta}\Big[t^8 - \tfrac{4}{3}(\kappa^2 - \kappa'^2)\,t^6 + \tfrac{2}{15}(4 - 19\kappa^2\kappa'^2)\,t^4$$
$$+ \tfrac{8}{15}(\kappa^2 - \kappa'^2)\,\kappa^2\kappa'^2 t^2 + \tfrac{1}{5}\kappa^4\kappa'^4\Big].$$

Now

$$3t^2 = \kappa^2 - \kappa'^2 \pm (1 - \kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$9t^4 = 2 - 5\kappa^2\kappa'^2 \pm 2(\kappa^2 - \kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$27t^6 = (4 - 7\kappa^2\kappa'^2)(\kappa^2 - \kappa'^2) \pm (4 - 13\kappa^2\kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$81t^8 = 8 - 40\kappa^2\kappa'^2 + 41\kappa^4\kappa'^4 \pm 4(2 - 5\kappa^2\kappa'^2)(\kappa^2 - \kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}}.$$

Whence on substitution, with $f^2g^2 = 1$,

$$I_2^s(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \frac{2^3}{3^4}\Big[(1 - \kappa^2\kappa'^2)^2 \pm (1 + \tfrac{1}{2}\kappa^2\kappa'^2)(\kappa^2 - \kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}}\Big] \quad . \quad (16).$$

The upper sign being taken for the zonal ($s = 0$), the lower for the sectorial harmonic ($s = 2$).

If these expressions be developed in powers of κ' as far as three terms of the series, I find, on re-introducing the factor f^2g^2,

$$I_2(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \frac{2^4}{5^4}(1 - 2\kappa'^2 + \tfrac{21}{16}\kappa'^4) \,.\, f^2g^2$$

$$= \frac{4\pi M}{5} \cdot \frac{2^4}{5^4}(1 - 4\beta + \tfrac{37}{4}\beta^2) \,.\, f^2g^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

$$I_2^2(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \tfrac{1}{3}\kappa'^4(1 - \kappa'^2 + \tfrac{11}{16}\kappa'^4) \,.\, f^2g^2$$

$$= \frac{4\pi M}{5} \cdot \tfrac{1}{3}\left(\frac{2\beta}{1 + \beta}\right)^2 (1 - 2\beta + \tfrac{19}{4}\beta^2) \,.\, f^2g^2 \quad . \quad . \quad . \quad . \quad (18).$$

In "Harmonics" I made the following definitions

$$\left.\begin{aligned} \mathfrak{P}_2(\mu) &= P_2(\mu) - \tfrac{1}{4}\beta P_2^2(\mu) = 1 - \tfrac{3}{2}(1 + \tfrac{1}{2}\beta)\cos^2\theta, \\ \mathfrak{C}_2(\phi) &= 1 - \tfrac{3}{2}\beta\cos 2\phi \qquad\quad\;\; = 1 - \tfrac{3}{2}\beta + \tfrac{3}{2}\beta\sin^2\phi \end{aligned}\right\} \quad . \quad . \quad . \quad (19).$$

In order to make the two forms of definition agree we must take

$$ft^2 = 1, \quad gt^2 = 1 - \tfrac{3}{2}\beta.$$

Thus

$$f^2g^2 = \frac{1}{t^8}(1 - 3\beta + \tfrac{9}{4}\beta^2).$$

Now on development

$$t^8 = (\tfrac{2}{3})^4(1 - 5\kappa'^2 + \tfrac{81}{8}\kappa'^4) = (\tfrac{2}{3})^4(1 - 10\beta + \tfrac{101}{2}\beta^2).$$

Whence

$$f^2g^2 = (\tfrac{3}{2})^4(1 + 7\beta + \tfrac{87}{4}\beta^2).$$

Introducing this I find

$$I_2(\cos) = \frac{4\pi M}{5}(1 + 3\beta + 3\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

agreeing with the result on p. 549 of "Harmonics" with $i = 2$, $s = 0$, type EEC.

Again in "Harmonics"

$$\left.\begin{aligned} \mathfrak{P}_2{}^2(\mu) &= 3\beta P_2(\mu) + P_2{}^2(\mu) = 3\beta + 3(1 - \tfrac{3}{2}\beta)\cos^2\theta, \\ \mathfrak{C}_2{}^2(\phi) &= \tfrac{1}{2}\beta + \cos 2\phi \qquad\quad = 1 + \tfrac{1}{2}\beta - 2\sin^2\phi \end{aligned}\right\} \quad . \quad . \quad (21).$$

In order to make the two definitions agree we must take

$$f\kappa^2 = -3(1 - \tfrac{3}{2}\beta), \qquad g\kappa'^2 = 2\,;$$

or

$$f = -3(1 + \tfrac{1}{2}\beta - \beta^2), \quad g = 2\left(\frac{1+\beta}{2\beta}\right).$$

So that $f^2g^2 = 2^2 \,.\, 3^2\left(\dfrac{1+\beta}{2\beta}\right)^2(1 + \beta - \tfrac{7}{4}\beta^2)$.

Introducing this in (18) we have

$$I_2{}^2(\cos) = \frac{4\pi}{7}M \,.\, 12(1 - \beta + \beta^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 2$, $s = 2$, type EEC.

(2) *The Cosine Tesseral Harmonic.*

These are defined thus,—

$$\left.\begin{aligned} \mathsf{P}_2{}^1(\mu) &= \sin\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}} = f\,.\,(\kappa^2 - \kappa^2\cos^2\theta)^{\frac{1}{2}}(\kappa'^2 + \kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathsf{C}_2{}^1(\phi) &= \cos\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} = g(\kappa^2 + \kappa'^2\sin^2\phi)^{\frac{1}{2}}(\kappa'^2 - \kappa'^2\sin^2\phi)^{\frac{1}{2}}, \end{aligned}\right\}. \quad (23)$$

where $f = \dfrac{1}{\kappa}$, $g = \dfrac{1}{\kappa'}$.

Squaring $\mathsf{P}_2{}^1$ we find

$$A_0 = \kappa^2\kappa'^2, \; A_1 = (\kappa^2 - \kappa'^2), \; A_2 = -1.$$

On substituting in the formula, I find, on putting $f^2g^2\kappa^2\kappa'^2 = 1$ and reducing,

$$I_2{}^1(\cos) = \frac{4\pi k^5\cos\beta\cos\gamma}{5\sin^3\beta}\,.\,\tfrac{1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

In "Harmonics" the definitions were

$$\left.\begin{aligned}\mathbf{P}_2^1(\mu) &= \left(\frac{\frac{1+\beta}{1-\beta}-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}} P_2^1(\mu) = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \sin\theta \left(1-\frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}} \\ \mathbf{C}_2^1(\phi) &= (1-\beta\cos 2\phi)^{\frac{1}{2}}\cos\phi = (1+\beta)^{\frac{1}{2}}\cos\phi\left(1-\frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}}\end{aligned}\right\} \quad (25).$$

In order to make the two definitions agree we must take

$$f\kappa = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \qquad g\kappa' = (1+\beta)^{\frac{1}{2}};$$

so that $f^2g^2\kappa^2\kappa'^2 = 3^2\,\dfrac{(1+\beta)^2}{1-\beta} = 3^2(1+3\beta+4\beta^2)$. On multiplying (24) by this factor, we have

$$I_2^1(\cos) = \frac{4\pi}{5}\,\mathrm{M}\,.\,3\,(1+3\beta+4\beta^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

agreeing with the result on p. 549 of "Harmonics" with $i = 2$, $s = 1$, type EOC.

(3) *The Tesseral Sine Harmonic.*

This is defined thus,—

$$\left.\begin{aligned}\mathfrak{P}_2^1(\mu) &= \sin\theta\cos\theta = f\,.\,\kappa\cos\theta\,(\kappa^2-\kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathbf{S}_2^1(\phi) &= \sin\phi\,(1-\kappa'^2\cos^2\phi)^{\frac{1}{2}} = g\,.\,\sqrt{-1}\,\kappa'\sin\phi\,(\kappa^2+\kappa'^2\sin^2\phi)^{\frac{1}{2}},\end{aligned}\right\}. \quad (27)$$

where $f = \dfrac{1}{\kappa^2}$, $g = \dfrac{1}{\kappa'\sqrt{-1}}$.

Squaring $\mathfrak{P}_2^1$ we find

$$A_0 = 0,\; A_1 = \kappa^2,\; A_2 = -1$$

whence, on putting $-f^2g^2\kappa^4\kappa'^2 = 1$,

$$I_2^1(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\,.\,\tfrac{1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

In "Harmonics" the definitions were

$$\left.\begin{aligned}\mathfrak{P}_2^1(\mu) &= P_2^1(\mu) = 3\sin\theta\cos\theta, \\ \mathbf{S}_2^1(\phi) &= \sin\phi\,(1-\beta\cos 2\phi)^{\frac{1}{2}} = (1+\beta)^{\frac{1}{2}}\sin\phi\left(1-\frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}}\end{aligned}\right\} \quad (29).$$

Therefore, to make the two definitions agree, we must take

$$f\kappa^2 = 3, \qquad g\kappa'\sqrt{-1} = (1+\beta)^{\frac{1}{2}}.$$

Therefore $-f^2g^2\kappa^4\kappa'^2 = 3^2(1+\beta)$, and on multiplying (28) by this factor we have

$$I_2^1(\sin) = \frac{4\pi}{5}\,\mathrm{M}\,.\,3(1+\beta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

agreeing with the result on p. 549 of "Harmonics" with $i = 2$, $s = 1$, type EOS.

(5) *The Sectorial Sine Harmonic.*

This is defined thus:—

$$\left.\begin{aligned} \mathsf{P}_2^2(\mu) &= \cos\theta\,(1-\kappa^2\sin^2\theta)^{\frac{1}{2}}, \\ \mathfrak{S}_2^2(\phi) &= \sin\phi\cos\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

If in the last integral we had written $\frac{1}{2}\pi - \theta$ for ϕ, and $\frac{1}{2}\pi - \phi$ for θ, and κ' for κ, $\mathfrak{P}_2^1$ would have become $\mathfrak{S}_2^2$, and S_2^1 would have become P_2^2. Therefore the result (28) gives what is needed by merely interchanging κ and κ'.

Therefore

$$I_2^2(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\,.\,\tfrac{1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

For the purpose of comparison I must put

$$\mathsf{P}_2^2(\mu) = f\,.\,\kappa\cos\theta\,(\kappa'^2+\kappa^2\cos^2\theta)^{\frac{1}{2}}, \quad \mathfrak{S}_2^2(\phi) = g\,.\,\sqrt{-1}\,\kappa'\sin\phi\,(\kappa'^2-\kappa'^2\sin^2\phi)^{\frac{1}{2}}, \quad (31)$$

and

$$I_2^2(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\left(-\tfrac{1}{3}f^2g^2\kappa^2\kappa'^4\right) \quad . \quad . \quad . \quad . \quad . \quad (33).$$

In "Harmonics" the definition was

$$\left.\begin{aligned} \mathsf{P}_2^2(\mu) &= \left(\frac{\frac{1+\beta}{1-\beta}-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}} P_2^2(\mu) = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\cos\theta\left(1-\frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}}, \\ \mathfrak{S}_2^2(\phi) &= \sin 2\phi \qquad\qquad\qquad = 2\sin\phi\cos\phi. \end{aligned}\right\} \,.\, (34).$$

In order to make the two definitions agree we must take

$$f\kappa = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \qquad g\sqrt{-1}\,.\,\kappa'^2 = 2.$$

Thus $-f^2g^2\kappa^2\kappa'^4 = 2^2\,.\,3^2\,\dfrac{1+\beta}{1-\beta}$; introducing this in (33) we have

$$I_2^2(\sin) = \frac{4\pi\mathrm{M}}{5}\,.\,12\left(\frac{1+\beta}{1-\beta}\right) = \frac{4\pi\mathrm{M}}{5}\,12\,(1+2\beta+2\beta^2). \quad . \quad . \quad . \quad (35)$$

agreeing with the result on p. 548 of "Harmonics" with $i = 2$, $s = 2$, type EES.

THE HARMONICS OF THE THIRD ORDER.

In these the only coefficients are A_0, A_1, A_2, A_3, and (2) becomes

$$I_3^s\binom{\cos}{\sin} = \frac{4\pi k^3 \cos\beta\cos\gamma}{\sin^3\beta}\{A_0{}^2 - \tfrac{1}{3}\kappa^2\kappa'^2 A_1{}^2 + \tfrac{1}{5}\kappa^4\kappa'^4 A_2{}^2 - \tfrac{1}{7}\kappa^6\kappa'^6 A_3{}^2$$
$$+ 2(\kappa^2 - \kappa'^2)[\tfrac{1}{3}A_0A_1 - \tfrac{2}{15}\kappa^2\kappa'^2 A_1A_2 + \tfrac{3}{35}\kappa^4\kappa'^4 A_2A_3]$$
$$+ \tfrac{2}{15}(4 - 9\kappa^2\kappa'^2)A_0A_2 - \tfrac{2}{105}(12 - 25\kappa^2\kappa'^2)\kappa^2\kappa'^2 A_1A_3$$
$$+ \tfrac{4}{35}(\kappa^2 - \kappa'^2)(4 - 5\kappa^2\kappa'^2)A_0A_3\} \quad (s = 0, 1, 2, 3).$$

(1) *and* (4) *The Zonal and Second Tesseral Cosine Harmonics.*

These are defined thus:—

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= \sin\theta(\kappa^2\sin^2\theta - q^2)\\ \mathfrak{C}_3^s(\phi) &= (q'^2 - \kappa'^2\cos^2\phi)(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}}, \quad (s = 0, 2)\end{aligned}\right\} \quad . \quad . \quad . \quad (36)$$

where $q^2 = \frac{2}{5}[1 + \kappa^2 \mp (1 - \frac{7}{4}\kappa^2 + \kappa^4)^{\frac{1}{2}}]$, with the upper sign for $s = 0$ and the lower for $s = 2$. Writing

$$t^2 = \kappa^2 - q^2 = q'^2 - \kappa'^2$$
$$= \tfrac{1}{5}[3\kappa^2 - 2 \pm (4 - 7\kappa^2 + 4\kappa^4)^{\frac{1}{2}}] = \tfrac{1}{5}[1 - 3\kappa'^2 \pm (1 - \kappa'^2 + 4\kappa'^4)^{\frac{1}{2}}]$$

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= f.(t^2 - \kappa^2\cos^2\theta)(\kappa^2 - \kappa^2\cos^2\theta)^{\frac{1}{2}},\\ \mathfrak{C}_3^s(\phi) &= g.(t^2 + \kappa'^2\sin^2\phi)(\kappa^2 + \kappa'^2\sin^2\phi)^{\frac{1}{2}},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where $f = \dfrac{1}{\kappa}$, $g = 1$.

Squaring $\mathfrak{P}_3^s$ we find

$$A_0 = t^4\kappa^2, \quad A_1 = -(2\kappa^2 + t^2)t^2, \quad A_2 = 2t^2 + \kappa^2, \quad A_3 = -1.$$

After some rather tedious reductions I find (for $s = 0, 2$)

$$I_3^s(\cos) = \frac{4\pi k^3\cos\beta\cos\gamma}{\sin^3\beta}\kappa^2\{\tfrac{1}{3}t^8 - \tfrac{4}{15}(1 - 3\kappa'^2)t^6 + \tfrac{2}{105}(4 - 25\kappa'^2 + 33\kappa'^4)t^4$$
$$+ \tfrac{4}{105}(2 - 5\kappa'^2)\kappa^2\kappa'^2 t^2 + \tfrac{1}{35}\kappa^4\kappa'^4\}f^2g^2.$$

Now writing $D = (1 - \kappa'^2 + 4\kappa'^4)^{\frac{1}{2}}$,

$5t^2 = 1 - 3\kappa'^2 \pm D,$

$5^2 t^4 = 2 - 7\kappa'^2 + 13\kappa'^4 \pm 2(1 - 3\kappa'^2)D,$

$5^3 t^6 = 4 - 21\kappa'^2 + 48\kappa'^4 - 63\kappa'^6 \pm (4 - 19\kappa'^2 + 31\kappa'^4)D,$

$5^4 t^8 = 8 - 56\kappa'^2 + 177\kappa'^4 - 314\kappa'^6 + 313\kappa'^8 \pm (8 - 52\kappa'^2 + 136\kappa'^4 - 156\kappa'^6)D.$

On substituting these in the above expression, and noting that $\kappa^2 f'^2 g^2$ will be unity with the definition adopted, I find

$$\left.\begin{aligned} I_3(\cos) &= \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \frac{2^3}{5^4} D\left[(1-\tfrac{1}{2}\kappa'^2)(1-\kappa'^2-\tfrac{8}{3}\kappa'^4)+(1-\kappa'^2+\tfrac{2}{3}\kappa'^4)D\right], \\ I_3^2(\cos) &= \text{the same with the sign of } D \text{ changed.} \end{aligned}\right\} \quad (38).$$

If these expressions be developed in powers of κ', and if the factor $\kappa^2 f'^2 g^2$ be re-introduced, I find

$$\begin{aligned} I_3(\cos) &= \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} (\tfrac{2}{5})^4 (1-2\kappa'^2+\tfrac{49}{16}\kappa'^4) \,.\, \kappa^2 f'^2 g^2. \\ &= \frac{4\pi}{7} \mathrm{M} (\tfrac{2}{5})^4 (1-4\beta+\tfrac{65}{4}\beta^2) \,.\, \kappa^2 f'^2 g^2 \\ I_3^2(\cos) &= \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \frac{1}{3\cdot 5}\kappa'^4 (1-\kappa'^2+\tfrac{31}{16}\kappa'^4) \,.\, \kappa^2 f'^2 g^2 \\ &= \frac{4\pi}{7} \mathrm{M} \,.\, \frac{1}{3\cdot 5}\kappa'^4 (1-2\beta+\tfrac{39}{4}\beta^2) \,.\, \kappa^2 f^2 g^2. \end{aligned}$$

In "Harmonics" I defined,

$$\mathfrak{P}_3(\mu) = P_3(\mu) - \tfrac{1}{4}\beta P_3^2(\mu) = \sin\theta\left[\tfrac{5}{2}\sin^2\theta(1+\tfrac{3}{2}\beta) - \tfrac{3}{2}(1+\tfrac{5}{2}\beta)\right]. \quad (39).$$

In order that our previous definition may agree with this we must have

$$f\kappa q^2 = \tfrac{3}{2}(1+\tfrac{5}{2}\beta), \quad f\kappa^3 = \tfrac{5}{2}(1+\tfrac{3}{2}\beta).$$

Now $$\begin{aligned} q^2 &= \tfrac{1}{5}\left[4-2\kappa'^2-\sqrt{(1-\kappa'^2+4\kappa'^4)}\right] = \tfrac{3}{5}\left[1-\tfrac{1}{2}\kappa'^2-\tfrac{5}{8}\kappa'^4\right] \\ &= \tfrac{3}{5}\left[1-\beta-\tfrac{3}{2}\beta^2\right] \end{aligned}$$

whence $f\kappa = \frac{5}{2}(1+\frac{7}{2}\beta+5\beta^2)$, and this value of $f\kappa$ satisfies the second equation.

With regard to $\mathsf{C}_3(\phi)$ there is a mistake in the table (the only one I have detected therein) on p. 556 of "Harmonics," for the coefficient of the second term should not be 3β but $\frac{5}{2}\beta$. The mistake obviously arose from my using the formula for p_2 instead of that for p'_2 as given on p. 490.

With the corrected coefficient the definition is

$$\left.\begin{aligned} \mathsf{C}_3(\phi) &= (1-\beta\cos 2\phi)^{\frac{1}{2}}(1-\tfrac{5}{2}\beta\cos 2\phi) \\ &= (1+\beta)^{\frac{1}{2}}(1+\tfrac{5}{2}\beta-5\beta\cos^2\phi)\left(1-\frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad (40).$$

In order that the previous definition may agree with this we must have

$$\begin{aligned} gq'^2 &= (1+\beta)^{\frac{1}{2}}(1+\tfrac{5}{2}\beta) = 1+3\beta+\tfrac{9}{8}\beta^2, \\ g\kappa'^2 &= 5\beta(1+\beta)^{\frac{1}{2}}. \end{aligned}$$

But $$q'^2 = 1 - q^2 = \tfrac{2}{5}(1 + \tfrac{3}{2}\beta + \tfrac{9}{4}\beta^2), \text{ and thence}$$

$$g = \tfrac{5}{2}(1 + \tfrac{3}{2}\beta - \tfrac{27}{8}\beta^2).$$

This value of g will be found to give the correct value for $g\kappa'^2$.

Then $$fg\kappa = (\tfrac{5}{2})^2(1 + 5\beta + \tfrac{55}{8}\beta^2),$$

and $$f^2g^2\kappa^2 = (\tfrac{5}{2})^4(1 + 10\beta + \tfrac{155}{4}\beta^2).$$

Introducing this into the value of $I_3(\cos)$, we find

$$I_3(\cos) = \frac{4\pi}{7}\mathrm{M}(1 + 6\beta + 15\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

agreeing with the result on p. 549 of "Harmonics" for $i = 3$, $s = 0$, type OEC.

Again in "Harmonics" I defined

$$\mathfrak{P}_3^2(\mu) = 15\beta P_3(\mu) + P_3^2(\mu) = 15\sin\theta[1 - \tfrac{3}{2}\beta - (1 - \tfrac{5}{2}\beta)\sin^2\theta] \quad . \quad (42).$$

To make the former definition agree with this we must take

$$f\kappa q^2 = -15(1 - \tfrac{3}{2}\beta), \quad f\kappa^3 = -15(1 - \tfrac{5}{2}\beta).$$

In the present case

$$q^2 = \tfrac{1}{5}[4 - 2\kappa'^2 + \sqrt{(1 - \kappa'^2 + 4\kappa'^4)}] = 1 - \tfrac{1}{2}\kappa'^2 + \tfrac{3}{8}\kappa'^4 + \tfrac{3}{16}\kappa'^6,$$
$$= 1 - \beta + \tfrac{5}{2}\beta^2 - \tfrac{5}{2}\beta^3.$$

Omitting the term in β^3 we find, with this value of q^2,

$$f\kappa = -15(1 - \tfrac{1}{2}\beta - 3\beta^2), \text{ and that the second equation is satisfied.}$$

Again I defined

$$\mathbf{C}_3^2(\phi) = (\tfrac{3}{2}\beta + \cos 2\phi)(1 - \beta\cos 2\phi)^{\frac{1}{2}} = (1+\beta)^{\frac{1}{2}}(2\cos^2\phi - 1 + \tfrac{3}{2}\beta)\left(1 - \frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}} (43).$$

Hence to secure agreement we must take

$$gq'^2 = -(1 - \tfrac{3}{2}\beta)(1 + \beta)^{\frac{1}{2}} = -(1 - \beta - \tfrac{7}{8}\beta^2),$$
$$g\kappa'^2 = -2(1 + \beta)^{\frac{1}{2}}.$$

Now $q'^2 = 1 - q^2 = \beta(1 - \tfrac{5}{2}\beta + \tfrac{5}{2}\beta^2)$, and therefore

$$g = -\frac{1}{\beta}(1 + \tfrac{3}{2}\beta + \tfrac{3}{8}\beta^2).$$

The second equation is satisfied.

We have then

$$f^2 g^2 \kappa^2 \kappa'^4 = 2^2 \,.\, 3^2 \,.\, 5^2 (1 + \beta)(1 - \beta - \tfrac{23}{4}\beta^2) = 2^2 \,.\, 3^2 \,.\, 5^2 (1 - 0 \,.\, \beta - \tfrac{27}{4}\beta^2).$$

Introducing this into the value of $I_3^2(\cos)$, we find

$$I_3^2(\cos) = \frac{4\pi M}{7} \,.\, 3 \,.\, 4 \,.\, 5 (1 - 2\beta + 3\beta^2) \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44),$$

agreeing with the result in p. 548 of "Harmonics" for $i = 3$, $s = 2$, type OEC.

(2) *and* (6) *First Tesseral Cosine Harmonic and Sectorial Cosine Harmonic.*

These are defined thus :—

$$\left.\begin{aligned} \mathsf{P}_3^s(\mu) &= (\kappa^2 \sin^2\theta - q^2)(1 - \kappa^2 \sin^2\theta)^{\frac{1}{2}}, \\ \mathfrak{C}_3^s(\phi) &= \cos\phi\,(q'^2 - \kappa'^2 \cos^2\phi), \quad (s = 1,\ 3) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45),$$

where $q^2 = \frac{1}{5}(1 + 2\kappa^2 \mp (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}})$, with upper sign for $s = 1$ and lower sign for $s = 3$, and $q'^2 = 1 - q^2$.

Writing $t'^2 = \kappa'^2 - q'^2$,

$$\left.\begin{aligned} \mathsf{P}_3^s(\mu) &= f(t'^2 + \kappa^2 \cos^2\theta)(\kappa'^2 + \kappa^2 \cos^2\theta)^{\frac{1}{2}}, \\ \mathfrak{C}_3^s(\phi) &= g(t'^2 - \kappa'^2 \sin^2\phi)(\kappa'^2 - \kappa'^2 \sin^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (46),$$

where $f = -1$, $g = -\dfrac{1}{\kappa'}$.

It is clear that $[\mathsf{P}_3^s(\mu)\mathfrak{C}_3^s(\phi)]^2 (s = 1, 3)$ has the same form as $[\mathfrak{P}_3^s(\mu)\mathsf{C}_3^s(\phi)]^2 (s = 0 . 2)$ when in the latter we interchange θ with $\frac{1}{2}\pi - \phi$, and κ with κ'. The interchange of the variables of integration clearly makes no difference in the result, and therefore we need only interchange κ and κ', and replace t by t'.

In the present instance

$$t'^2 = \kappa'^2 - q'^2 = q^2 - \kappa^2 = \tfrac{1}{5}[1 - 3\kappa^2 \mp (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}}].$$

This shows that t'^2 is the same function of κ^2 that t^2 was of κ'^2, but that $I_3^1(\cos)$ is analogous with $I_3^2(\cos)$, and $I_3^3(\cos)$ with $I_3(\cos)$. Thus we may at once write down the results by interchanging κ and κ' throughout.

Let $$D' = (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}}.$$

Then putting $\kappa'^2 f^2 g^2 = 1$, we have by symmetry with (38)

$$\left.\begin{aligned} I_3^1(\cos) &= \frac{4\pi k^3 \cos\beta \cos\gamma}{7 \sin^3\beta} \,.\, \frac{2^3}{5^4} D'[(1 - \kappa^2 + \tfrac{2}{3}\kappa^4)D' - (1 - \tfrac{1}{2}\kappa^2)(1 - \kappa^2 - \tfrac{8}{3}\kappa^4)], \\ I_3^3(\cos) &= \text{the same with the sign of } D' \text{ changed.} \end{aligned}\right\} (47).$$

If these expressions be developed in powers of κ' I find, on reintroducing the factor $\kappa'^2 f^2 g^2$,

$$I_3^1(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{7\sin^3\beta} \cdot \frac{2^7}{3.5^4} \cdot (1 - \tfrac{9}{4}\kappa'^2 + \tfrac{361}{256}\kappa'^4) \cdot \kappa'^2 f^2 g^2,$$

$$= \frac{4\pi}{7} \mathrm{M} \cdot \frac{2^7}{3.5^4} (1 - \tfrac{9}{2}\beta + \tfrac{649}{64}\beta^2) \cdot \kappa'^2 f^2 g^2,$$

$$I_3^3(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{7\sin^3\beta} \cdot \tfrac{1}{2.5}\kappa'^4 (1 - \tfrac{3}{2}\kappa'^2 + \tfrac{183}{256}\kappa'^4) \cdot \kappa'^2 f^2 g^2,$$

$$= \frac{4\pi}{7} \mathrm{M} \cdot \tfrac{1}{2.5} \left(\frac{2\beta}{1+\beta}\right)^2 (1 - 3\beta + \tfrac{375}{64}\beta^2) \cdot \kappa'^2 f^2 g^2.$$

In "Harmonics" I defined

$$\mathsf{P}_3^1(\mu) = \left(\frac{\frac{1+\beta}{1-\beta} - \mu^2}{1-\mu^2}\right)^{\frac{1}{2}} [P_3^1(\mu) - \tfrac{3}{16}\beta(1 + \tfrac{3}{4}\beta) P_3^3(\mu)],$$

$$= \tfrac{15}{2}\left(1 - \frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}} \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} [\sin^2\theta(1 + \tfrac{3}{8}\beta + \tfrac{9}{32}\beta^2) - \tfrac{1}{5}(1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2)] \quad (48).$$

But we have defined it above by

$$\mathsf{P}_3^1(\mu) = f(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}} (\kappa^2\sin^2\theta - q^2).$$

Therefore

$$f\kappa^2 = \tfrac{15}{2}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} (1 + \tfrac{3}{8}\beta + \tfrac{9}{32}\beta^2),$$

$$fq^2 = \tfrac{3}{2}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} (1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2).$$

Now $q^2 = \frac{1}{5}(1 - \frac{1}{4}\kappa'^2 - \frac{15}{64}\kappa'^4) = \frac{1}{5}(1 - \frac{1}{2}\beta - \frac{7}{16}\beta^2)$.

Whence $f = \frac{15}{2}(1 + \frac{27}{8}\beta + \frac{189}{32}\beta^2)$.

This value also satisfies the expression for $f\kappa^2$.

Again I defined

$$\mathfrak{C}_3^1(\phi) = \cos\phi - \tfrac{5}{8}\beta(1 + \tfrac{3}{4}\beta)\cos 3\phi,$$

$$= \cos\phi[1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2 - \tfrac{5}{2}\beta(1 + \tfrac{3}{4}\beta)\cos^2\phi] \quad . \quad . \quad . \quad (49).$$

But we have defined it above by

$$\mathfrak{C}_3^1 = g\kappa'\cos\phi(\kappa'^2\cos^2\phi - q'^2).$$

Therefore

$$g\kappa' q'^2 = -(1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2), \qquad g\kappa'^3 = -\tfrac{5}{2}\beta(1 + \tfrac{3}{4}\beta^2).$$

With the above value for q^2 we have $q'^2 = \frac{4}{5}(1 + \frac{1}{8}\beta + \frac{7}{64}\beta^2)$; whence

$$g\kappa' = -\tfrac{5}{4}(1 + \tfrac{7}{4}\beta + \tfrac{69}{64}\beta^2).$$

Therefore $f^2 g^2 \kappa'^2 = \frac{3^2 . 5^4}{2^6} [1 + \frac{41}{4} \beta + \frac{3331}{64} \beta^2]$, and I find

$$I_3^1 (\cos) = \frac{4\pi}{7} \mathrm{M} . 6 (1 + \tfrac{23}{4} \beta + \tfrac{257}{16} \beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (50),$$

agreeing with the result on p. 549 of "Harmonics" with $i = 3$, $s = 1$, type OOC.

In "Harmonics" I defined

$$\mathsf{P}_3^3 (\mu) = \left(\frac{\frac{1+\beta}{1-\beta} - \mu^2}{1 - \mu^2} \right)^{\frac{1}{2}} [\tfrac{5}{4} \beta (1 + \tfrac{3}{4} \beta) P_3^1 (\mu) + P_3^3 (\mu)],$$

$$= 15 \left(1 - \frac{1-\beta}{1+\beta} \sin^2 \theta \right)^{\frac{1}{2}} [1 + \tfrac{7}{8} \beta + \tfrac{9}{32} \beta^2 - \sin^2 \theta (1 + \tfrac{3}{8} \beta - \tfrac{19}{32} \beta^2)] \quad . \quad (51).$$

But $\mathsf{P}_3^3 (\mu) = f (1 - \kappa^2 \sin^2 \theta)^{\frac{1}{2}} (\kappa^2 \sin^2 \theta - q^2)$.

Therefore $f\kappa^2 = - 15 (1 + \frac{3}{8} \beta - \frac{19}{32} \beta^2)$, $\quad fq^2 = - 15 (1 + \frac{7}{8} \beta + \frac{9}{32} \beta^2)$.

Now $$q^2 = 1 - \frac{3}{2^2} \kappa'^2 + \frac{3}{2^6} \kappa'^4 + \frac{21}{2^9} \kappa'^6 \ldots$$

Therefore $$q^2 = 1 - \tfrac{3}{2} \beta + \tfrac{27}{16} \beta^2, \text{ and}$$

$$f = - 15 (1 + \tfrac{19}{8} \beta + \tfrac{69}{32} \beta^2).$$

This also gives the correct value to $f\kappa^2$.

Again $$\mathfrak{C}_3^3 (\phi) = \tfrac{3}{8} \beta (1 + \tfrac{3}{4} \beta) \cos \phi + \cos 3\phi,$$

$$= \cos \phi [4 \cos^2 \phi - 3 (1 - \tfrac{1}{8} \beta - \tfrac{3}{32} \beta^2)] \quad . \quad . \quad . \quad . \quad . \qquad (52).$$

But $\mathfrak{C}_3^3 (\phi) = g\kappa' \cos \phi (\kappa'^2 \cos^2 \phi - q'^2)$.

Therefore $g\kappa' = \frac{4}{\kappa'^2}$, and $g\kappa' . q'^2 = 3 (1 - \frac{1}{8} \beta - \frac{3}{32} \beta^2)$.

If we eliminate $g\kappa'$, these equations give the correct value for q'^2.

Then $fg\kappa' = - \frac{60}{\kappa'^2} (1 + \frac{19}{8} \beta + \frac{69}{32} \beta^2)$.

Therefore

$$f^2 g^2 \kappa'^2 = \frac{3^2 . 4^2 . 5^2}{\kappa'^4} (1 + \tfrac{19}{4} \beta + \tfrac{637}{64} \beta^2).$$

Hence we find

$$I_3^3 (\cos) = \frac{4\pi \mathrm{M}}{7} . 360 (1 + \tfrac{7}{4} \beta + \tfrac{25}{16} \beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 3$, type OOC.

(3) *and* (7) *First Tesseral Sine Harmonic and Sectorial Sine Harmonic.*

These are defined thus :—

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= \cos\theta\,(\kappa^2\sin^2\theta - q^2),\\ \mathfrak{S}_3^s(\phi) &= \sin\phi\,(q'^2 - \kappa'^2\cos^2\phi), \quad (s = 1,\ 3),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

where $q^2 = \frac{1}{5}[2 + \kappa^2 \mp (4 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$, with the upper sign for $s = 1$, the lower for $s = 3$, and $q'^2 = 1 - q^2$.

Writing $$t^2 = \kappa^2 - q^2,$$

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= f\,.\,\kappa\cos\theta\,(t^2 - \kappa^2\cos^2\theta),\\ \mathfrak{S}_3^s(\phi) &= g\,.\,\kappa'\sqrt{-1}\,\sin\phi\,(t^2 + \kappa'^2\sin^2\phi),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

where $f = \dfrac{1}{\kappa}$, $g = \dfrac{1}{\kappa'\sqrt{-1}}$.

Squaring $\mathfrak{P}_3^s$ we find,

$$A_0 = 0, \quad A_1 = t^4, \quad A_2 = -2t^2, \quad A_3 = 1.$$

On substitution in the formula for harmonics of the third order I find

$$I_3^s(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{\sin^3\beta}\left[\tfrac{1}{3}t^8 - \tfrac{8}{15}(\kappa^2 - \kappa'^2)\,t^6 + \tfrac{2}{105}(12 - 67\kappa^2\kappa'^2)\,t^4 \right.$$
$$\left. + \tfrac{12}{35}(\kappa^2 - \kappa'^2)\,\kappa^2\kappa'^2t^2 + \tfrac{1}{7}\,\kappa^4\kappa'^4\right](-f^2g^2\kappa^2\kappa'^2).$$

If we write

$$D = (1 - \tfrac{1}{4}\,\kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$\frac{5}{2}\,t^2 = 1 - 2\kappa'^2 \pm D,$$

$$\frac{5^2}{2^2}\,t^4 = 2 - \tfrac{17}{4}\,\kappa^2\kappa'^2 \pm 2\,(1 - 2\kappa'^2)\,D,$$

$$\frac{5^3}{2^3}\,t^6 = 4 - \tfrac{51}{4}\,\kappa^2\kappa'^2 - \tfrac{19}{2}\,\kappa'^6 \pm (4 - \tfrac{49}{4}\,\kappa^2\kappa'^2)\,D,$$

$$\frac{5^4}{2^4}\,t^8 = 8 - 34\kappa'^2 + \tfrac{897}{16}\,\kappa'^4 - \tfrac{353}{8}\,\kappa'^6 + \tfrac{353}{16}\,\kappa'^8 \pm (8 - 25\kappa'^2 + 51\kappa'^4 - 34\kappa'^6)\,D.$$

On substitution I find, on putting $-f^2g^2\kappa^2\kappa'^2 = 1$,

$$\left.\begin{aligned}I_3^1(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{7\sin^3\beta}\,.\,\frac{2^3}{3\,.\,5^4}\,&[8 - 14\kappa^2\kappa'^2 + 3\kappa^4\kappa'^4\\ &+ (\kappa^2 - \kappa'^2)(8 + 3\kappa^2\kappa'^2)(1 - \tfrac{1}{4}\,\kappa^2\kappa'^2)^{\frac{1}{2}}]\\ I_3^3(\sin) = \text{the same with the sign of the square root reversed.}&\end{aligned}\right\} \quad . \quad (56)$$

Developing these expressions in powers of κ'^2, reintroducing the factor $-f^2g^2\kappa^2\kappa'^2$, and reverting to the notation of "Harmonics," I find

$$I_3^1(\sin) = \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \frac{2^7}{3\,.\,5^4}(1 - \tfrac{7}{4}\kappa'^2 + \tfrac{169}{256}\kappa'^4)(-f^2g^2\kappa^2\kappa'^2)$$

$$= \frac{4\pi \mathrm{M}}{7} \cdot \frac{2^7}{3\,.\,5^4}(1 - \tfrac{7}{2}\beta + \tfrac{393}{64}\beta^2)(-f^2g^2\kappa^2\kappa'^2)$$

$$I_3^3(\sin) = \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \tfrac{1}{25}\kappa'^4(1 - \tfrac{1}{2}\kappa'^2 + \tfrac{55}{256}\kappa'^4)(-f^2g^2\kappa^2\kappa'^2)$$

$$= \frac{4\pi \mathrm{M}}{7} \cdot \tfrac{1}{25}\kappa'^4(1 - \beta + \tfrac{119}{64}\beta^2)(-f^2g^2\kappa^2\kappa'^2).$$

In "Harmonics" I defined

$$\mathfrak{P}_3^1(\mu) = P_3^1(\mu) - \tfrac{1}{16}\beta(1 - \tfrac{3}{4}\beta)P_3^3(\mu)$$
$$= \tfrac{15}{2}\cos\theta[\sin^2\theta(1 + \tfrac{1}{8}\beta - \tfrac{3}{32}\beta^2) - \tfrac{1}{5}(1 + \tfrac{5}{8}\beta - \tfrac{15}{32}\beta^2)] \quad . \quad . \quad (57).$$

To make our former definition agree with this we must take

$$f\kappa\,.\,\kappa^2 = \tfrac{15}{2}(1 + \tfrac{1}{8}\beta - \tfrac{3}{32}\beta^2); \qquad f\kappa\,.\,q^2 = \tfrac{3}{2}(1 + \tfrac{5}{8}\beta - \tfrac{15}{32}\beta^2).$$

Hence $f\kappa = \tfrac{15}{2}(1 + \tfrac{17}{8}\beta + \tfrac{69}{32}\beta^2)$.
It will be found that $q^2 = \tfrac{1}{5}(1 - \tfrac{3}{2}\beta + \tfrac{9}{16}\beta^2)$, and that $f\kappa\,.\,q^2$ has the above form.
Again I defined

$$\mathfrak{S}_3^1(\phi) = \sin\phi - \tfrac{5}{8}\beta(1 - \tfrac{3}{4}\beta)\sin 3\phi,$$
$$= \sin\phi[1 - \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2 + \tfrac{5}{2}\beta(1 - \tfrac{3}{4}\beta)\sin^2\phi] \quad . \quad . \quad . \quad (58).$$

To make our former definition agree with this we must take

$$g\kappa'\sqrt{-1}\,.\,t^2 = 1 - \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2,$$

$$g\kappa'\sqrt{-1}\,.\,\kappa'^2 = \tfrac{5}{2}\beta(1 - \tfrac{3}{4}\beta).$$

It will be found that $t^2 = \tfrac{4}{5}(1 - \tfrac{17}{8}\beta + \tfrac{151}{64}\beta^2)$.
Whence $g\kappa'\sqrt{-1} = \tfrac{5}{4}(1 + \tfrac{1}{4}\beta - \tfrac{27}{64}\beta^2)$, and $g\kappa'\sqrt{-1}\,.\,\kappa'^2$ has the correct form.
Therefore

$$fg\kappa\kappa'\sqrt{-1} = \frac{3\,.\,5^2}{2^3}(1 + \tfrac{19}{8}\beta + \tfrac{145}{64}\beta^2)$$

$$-f^2g^2\kappa^2\kappa'^2 = \frac{3^2\,.\,5^4}{2^6}(1 + \tfrac{19}{4}\beta + \tfrac{651}{64}\beta^2).$$

Whence

$$I_3^1(\sin) = \frac{4\pi}{7}\mathrm{M}\,.\,6(1 + \tfrac{5}{4}\beta - \tfrac{5}{16}\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 1$, type OOS.

In "Harmonics" I defined

$$\mathfrak{P}_3^3(\mu) = \tfrac{15}{4}\beta\left(1 - \tfrac{3}{4}\beta\right) P_3^1(\mu) + P_3^3(\mu),$$
$$= 15\cos\theta\left[-\left(1 - \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2\right)\sin^2\theta + 1 - \tfrac{3}{8}\beta + \tfrac{9}{32}\beta^2\right] \quad . \quad (60).$$

In order that this may agree with our former definition we must take

$$f\kappa\,.\,\kappa^2 = -15\left(1 - \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2\right), \qquad f\kappa\,.\,q^2 = -15\left(1 - \tfrac{3}{8}\beta + \tfrac{9}{32}\beta^2\right).$$

Whence $f\kappa = -15\left(1 + \frac{1}{8}\beta - \frac{11}{32}\beta^2\right)$.

It will be found that

$$q^2 = 1 - \tfrac{1}{4}\kappa'^2 + \tfrac{3}{64}\kappa'^4 + \tfrac{3}{512}\kappa'^6 + \dots$$
$$= 1 - \tfrac{1}{2}\beta + \tfrac{11}{16}\beta^2,$$

so that $f\kappa\,.\,q^2$ has the correct form.

Again I defined

$$\mathfrak{S}_3^3(\phi) = \tfrac{3}{8}\beta\left(1 - \tfrac{3}{4}\beta\right)\sin\phi + \sin 3\phi,$$
$$= 3\sin\phi\left[1 + \tfrac{1}{8}\beta - \tfrac{3}{32}\beta^2 - \tfrac{4}{3}\sin^2\phi\right] \quad . \quad . \quad . \quad . \quad . \quad (61).$$

In order to make our former definition agree with this we must take

$$g\kappa'\sqrt{-1}\,.\,t^2 = 3\left(1 + \tfrac{1}{8}\beta - \tfrac{3}{32}\beta^2\right), \qquad g\kappa'\sqrt{-1}\,.\,\kappa'^2 = -4.$$

Therefore $g\kappa'\sqrt{-1} = -\dfrac{4}{\kappa'^2}$.

It will be found that $t^2 = -\frac{3}{4}\kappa'^2\left(1 + \frac{1}{8}\beta - \frac{3}{32}\beta^2\right)$, so that $g\kappa'\sqrt{-1}\,.\,t^2$ has the correct form.

Then

$$fg\kappa\kappa'\sqrt{-1} = -\frac{3\,.\,4\,.\,5}{\kappa'^2}\left(1 + \tfrac{1}{8}\beta - \tfrac{11}{32}\beta^2\right),$$

and

$$-f^2g^2\kappa^2\kappa'^2 = \frac{3^2\,.\,4^2\,.\,5^2}{\kappa'^4}\left(1 + \tfrac{1}{4}\beta - \tfrac{43}{64}\beta^2\right).$$

Whence

$$I_3^3(\sin) = \frac{4\pi\mathrm{M}}{7}\,.\,360\left[1 - \tfrac{3}{4}\beta + \tfrac{15}{16}\beta^2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (62),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 3$, type OOS.

(5) *The Second Tesseral Sine Harmonic.*

This is defined thus :—

$$\left.\begin{aligned}\mathsf{P}_3^2(\mu) &= \sin\theta\cos\theta\left(1-\kappa^2\sin^2\theta\right)^{\frac{1}{2}} = f\,.\,\kappa\cos\theta\left(\kappa^2-\kappa^2\cos^2\theta\right)^{\frac{1}{2}}\left(\kappa'^2+\kappa^2\cos^2\theta\right)^{\frac{1}{2}},\\ \mathsf{S}_3^2(\phi) &= \sin\phi\cos\phi\left(1-\kappa'^2\cos^2\phi\right)^{\frac{1}{2}} = g\,.\,\kappa'\sqrt{-1}\sin\phi\left(\kappa^2+\kappa'^2\sin^2\phi\right)^{\frac{1}{2}}\left(\kappa'^2-\kappa'^2\sin^2\phi\right)^{\frac{1}{2}},\end{aligned}\right\}(63)$$

where $f = \dfrac{1}{\kappa^2}$, $g = \dfrac{1}{\kappa'^2\sqrt{-1}}$.

Squaring P_3^2 we have

$$A_0 = 0, \quad A_1 = \kappa^2\kappa'^2, \quad A_2 = \kappa^2 - \kappa'^2, \quad A_3 = -1.$$

Therefore

$$I_3^2(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{\sin^3\beta}\Big[-\tfrac{1}{3}\kappa^6\kappa'^6 + \tfrac{1}{5}(\kappa^2 - \kappa'^2)^2\kappa^4\kappa'^4 - \tfrac{1}{7}\kappa^6\kappa'^6$$
$$- \tfrac{4}{15}(\kappa^2 - \kappa'^2)^2\kappa^4\kappa'^4 - \tfrac{6}{35}(\kappa^2 - \kappa'^2)^2\kappa^4\kappa'^4$$
$$+ \tfrac{2}{105}(12 - 25\,\kappa^2\kappa'^2)\,\kappa^4\kappa'^4\Big].f^2g^2.$$

Reducing this expression and putting $-f^2g^2\kappa^4\kappa'^4 = 1$, we have

$$I_3^2(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta}\cdot\frac{1}{35} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

In "Harmonics" I defined

$$\mathsf{P}_3^2(\mu) = \left(\frac{\frac{1+\beta}{1-\beta} - \mu^2}{1-\mu^2}\right)^{\frac{1}{2}} P_3^2(\mu) = 15\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(1 - \frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}}\cos\theta\sin\theta. \quad . \quad (65).$$

To make our former definition agree with this we must take

$$f\kappa^2 = 15\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$$

Again I defined

$$\mathsf{S}_3^2(\phi) = (1 - \beta\cos 2\phi)^{\frac{1}{2}}\sin 2\phi = 2(1+\beta)^{\frac{1}{2}}\left(1 - \frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}}\sin\phi\cos\phi. \quad (66).$$

To make the former definition agree with this we must take

$$g\kappa'^2\sqrt{-1} = 2(1+\beta)^{\frac{1}{2}}.$$

Therefore

$$fg\kappa^2\kappa'^2\sqrt{-1} = 2.3.5\frac{(1+\beta)}{(1-\beta)^{\frac{1}{2}}}, \text{ and}$$

$$-f^2g^2\kappa^4\kappa'^4 = 2^2.3^2.5^2\frac{(1+\beta)^2}{1-\beta} = 2^2.3^2.5^2(1 + 3\beta + 4\beta^2).$$

Hence in the notation of "Harmonics"

$$I_3^2(\sin) = \frac{4\pi\mathrm{M}}{7}.3.4.5(1 + 3\beta + 4\beta^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 2$, type OES.

It may be convenient, as furnishing a kind of index to the foregoing investigation, to state that the $1 + 3 + 5 + 7$ integrals for the harmonics of orders 0, 1, 2, 3 are given in equations 3, 5, 9, 13, 16, 24, 28, 32, 38, 47, 56, 64, corresponding to the definitions contained in 4, 8, 12, 15, 23, 27, 31, 36, 45, 54, 63.

The definitions of the harmonic functions as given in my paper on Harmonic Analysis are repeated in 6, 10, 19, 21, 25, 29, 34, 39, 40, 42, 43, 48, 49, 51, 52, 57, 58, 60, 61, 65, 66. Corresponding to these latter definitions the approximate integrals are given in 7, 11, 14, 20, 22, 26, 30, 35, 41, 44, 50, 53, 59, 62, 67; and the results confirm the correctness of the general approximate formulæ for the integrals given in § 22 of the paper on Harmonic Analysis.

A mistake in that paper was detected in the value of the cosine-function for the third zonal harmonic, and the corrected value is given in (40).

It must be obvious that the method exhibited here may be applied to higher harmonics with whatever degree of accuracy is desired; but it is also clear that the labour of evaluating the integrals increases very much as they rise in order. It is probable that the approximate results of the previous paper will suffice for most practical applications.

APPENDIX.

On the Symmetry of the Cosine and Sine-functions with the P-*functions.*

In my previous papers I failed to notice that the symmetry between the P-functions and the cosine and sine-functions is not destroyed, but is only masked, in the approximate expressions for the harmonic functions.

For example, (39) shows us that

$$\mathfrak{P}_3(\mu) = P_3(\mu) - \tfrac{1}{4}\beta P_3^2(\mu),$$

therefore, in consequence of the symmetry which subsists, we ought to find

$$\mathbf{C}_3(\phi) = P_3\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right] - \tfrac{1}{4}\beta P_3^2\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right].$$

Now
$$P_3\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right] = \frac{(1-\beta\cos 2\phi)^{\frac{1}{2}}}{(1-\beta)^{3/2}}(1 + \tfrac{3}{2}\beta - \tfrac{5}{2}\beta\cos 2\phi),$$

$$P_3^2\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right] = -15\,\frac{(1-\beta\cos 2\phi)^{\frac{1}{2}}}{(1-\beta)^{3/2}}\,\beta(1-\cos 2\phi).$$

Whence
$$\mathbf{C}_3(\phi) = \frac{(1 + \frac{3}{2}\beta + \frac{15}{4}\beta^2)}{(1-\beta)^{3/2}}(1-\beta\cos 2\phi)^{\frac{1}{2}}(1 - \tfrac{5}{2}\beta\cos 2\phi).$$

This only differs by a constant factor from the expression (40).

It would be possible then to have only one type of function, viz. $\mathfrak{P}$ or $\mathbf{P}$, and to express all the cosine and sine-functions by means of the appropriate one of them. This would be found to be equivalent to expressing the latter functions in terms of powers of sin ϕ. For the purposes of practical application I do not think this would be so convenient as the use of cosines and sines of multiples of ϕ, and the advantage of using only one type of function would not compensate for the loss of convenience in the result. Accordingly I do not think it worth while to undertake the very laborious task of revising all the analysis of "Harmonics" from this point of view.

I may mention, however, that I have gone far enough in this direction to feel pretty confident that, if this new form of developing the cosine and sine-functions were adopted, the remarkable coincidence. mentioned in the footnote on p. 547 of "Harmonics," as to the form of the integrals of the squares of surface harmonics would become explicable.

POSTSCRIPT.

[*December* 24*th*, 1903.]

Mr. HOBSON has shown me how these integrals may be evaluated by a simpler method of analysis, without the intervention of elliptic integrals. As an example of the method he suggests I take the integral I_2 (cos) evaluated above.

The solid ellipsoidal harmonics are given, except as regards a factor, in § 3 of "The Pear-Shaped Figure."

In (19) of that paper we find

$$S_2 = \mathfrak{P}_2(\nu)\,\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi) = A\left[q^2x^2 + (1-2q^2)\,y^2 - q'^2z^2 + \frac{k^2}{\kappa^2}q^2q'^2\right],$$

where A is the factor to be evaluated so as to agree with the definitions

$$\mathfrak{P}_2(\nu) = \kappa^2\nu^2 - q^2, \qquad \mathfrak{P}_2(\mu) = \kappa^2\mu^2 - q^2, \qquad \mathfrak{C}_2(\phi) = q'^2 - \kappa'^2\cos^2\phi.$$

The ellipsoid over which we desire to integrate is defined by $\nu = \dfrac{1}{\kappa\sin\gamma}$, and the extremity of the c axis is defined by $\mu = \sin\theta = 1$, $\phi = \frac{1}{2}\pi$.

Hence at this point

$$S_2 = (\operatorname{cosec}^2\gamma - q^2)\,(\kappa^2 - q^2)\,q'^2.$$

But at the extremity of the c axis

$$x = 0, \qquad y = 0, \qquad z = c = \frac{k}{\kappa\sin\gamma}.$$

Therefore

$$(\operatorname{cosec}^2\gamma - q^2)\,(\kappa^2 - q^2)\,q'^2 = k^2A\left(-\frac{q'^2}{\kappa^2\sin^2\gamma} + \frac{q^2q'^2}{\kappa^2}\right) = -A\frac{k^2q'^2}{\kappa^2}(\operatorname{cosec}^2\gamma - q^2).$$

Therefore $A = -\dfrac{\kappa^2(\kappa^2 - q^2)}{k^2}$, and

$$S_2 = (\kappa^2 - q^2)\left[-q^2\kappa^2\frac{x^2}{k^2} - (1-2q^2)\,\kappa^2\frac{y^2}{k^2} + q'^2\kappa^2\frac{z^2}{k^2} - q^2q'^2\right].$$

Let us assume

$$x = a\xi = \frac{k\cos\gamma}{\kappa\sin\gamma}\,\xi, \qquad y = b\xi = \frac{k\cos\beta}{\kappa\sin\gamma}\,\eta, \qquad z = c\zeta = \frac{k}{\kappa\sin\gamma}\,\zeta.$$

Then when x, y, z is on the ellipsoid we have

$$\xi^2 + \eta^2 + \zeta^2 = 1.$$

Thus we may regard ξ, η, ζ as the coordinates of a point on a sphere of unit radius, or as direction cosines, if it is more convenient to do so. On substituting for x, y, z their values in terms of ξ, η, ζ we find

$$S_2 = (\operatorname{cosec}^2\gamma - q^2)(\kappa^2 - q^2)[-q^2\xi^2 - (1-2q^2)\eta^2 + q'^2\zeta^2].$$

On performing the same operation to the points on the boundary of an element $d\sigma$ of surface of the ellipsoid, we find

$$p\,d\sigma = abc\,d\omega = \frac{k^3\cos\beta\cos\gamma}{\sin^3\beta}\,d\omega,$$

where $d\omega$ is an element of the surface of the sphere of unit radius, or an element of solid angle.

Since on the surface of the ellipsoid $\mathfrak{P}_2(\nu) = \operatorname{cosec}^2\gamma - q^2$, it follows that

$$\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi) = (\kappa^2 - q^2)[-q^2\xi^2 - (1-2q^2)\eta^2 + q'^2\zeta^2].$$

Hence

$$I_2(\cos) = \frac{k^3\cos\beta\cos\gamma}{\sin^3\beta}(\kappa^2 - q^2)^2\int[-q^2\xi^2 - (1-2q^2)\eta^2 + q'^2\zeta^2]^2\,d\omega.$$

It is easy to prove that

$$\int\xi^4\,d\omega = \int\eta^4\,d\omega = \int\zeta^4\,d\omega = \frac{4\pi}{5},$$

$$\int\eta^2\zeta^2\,d\omega = \int\zeta^2\xi^2\,d\omega = \int\xi^2\eta^2\,d\omega = \frac{4\pi}{15}.$$

Therefore

$$\begin{aligned} I_2(\cos) &= \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\cdot\tfrac{1}{3}(\kappa^2 - q^2)^2[3q^4 + 3(1-2q^2)^2 + 3q'^4 - 2q'^2(1-2q^2) \\ &\qquad\qquad + 2q^2(1-2q^2) - 2q^2q'^2] \\ &= \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\cdot\tfrac{4}{3}(\kappa^2 - q^2)^2(1 - 3q^2q'^2). \end{aligned}$$

On substituting for q^2 its value, viz., $\frac{1}{3}(1 + \kappa^2 - (1 - \kappa^2\kappa'^2)^{\frac{1}{2}})$, and effecting reductions we arrive at the result given in (16) above.

It is obvious that this process is considerably simpler and more elegant from the point of view of theory, but to carry these operations through for all the integrals given above would entail a good deal of algebra. I think indeed that the work might not be very much less than what I have already done.

Mr. Hobson has further remarked that all the integrations may be avoided by the following theorem :—

If $F_n(\xi, \eta, \zeta)$ be a solid spherical harmonic function of ξ, η, ζ of degree n,

$$\int [F_n(\xi, \eta, \zeta)]^2 d\omega = 4\pi \frac{2^n . n!}{2n+1\,!} F_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) F_n(\xi, \eta, \zeta).$$

Considering, however, how simple are the integrals involved in his first method, it may be doubted whether this would save trouble.

INDEX SLIP.

Tilden, W. A.—The Specific Heats of Metals and the Relation of Specific Heat to Atomic Weight.—Part III.
Phil. Trans., A, vol. 203, 1904, pp. 139–149.

Specific Heats of various Metals; Changes with Temperature; Relation to Atomic Weights; of Compounds and Alloys; of Gaseous Carbon; Neumann's Law verified.
Tilden, W. A. Phil. Trans., A, vol. 203, 1904, pp. 139–149.

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VI. *The Specific Heats of Metals and the Relation of Specific Heat to Atomic Weight.—Part* III.

By W. A. Tilden, *D.Sc., F.R.S., Professor of Chemistry in the Royal College of Science, London.*

Received March 9,—Read March 17, 1904.

The law of Neumann assumes that when an atom enters into chemical combination it retains the same capacity for heat as when in the uncombined or elemental state. This generalisation is, however, based on the values observed for the mean specific heats of elements and their compounds between 0° and 100° C.*

Attention was directed in Part II. of this investigation† to the great differences found in the influence of temperature on the specific heats of various metals, such as aluminium on the one hand, and silver or platinum on the other. The experiments now about to be described were undertaken with the object of ascertaining to what extent these differences persist in the compounds of such elements.

If the calculated atomic heats of elements in combination are equal at various temperatures through a considerable range to the sum of the atomic heats of the same in the separate state at the same temperatures, then the difference between any two must be due to a fundamental difference in the atoms of the elements concerned and not to a difference in their states of aggregation when separate.

The specific heats of nickel sulphide and silver sulphide have already been compared through a range of temperature from − 182° to + 324° C., but owing to the low melting-point of sulphur, and especially to the occurrence of several allotropic modifications of this element in the solid state, it was not thought worth while to attempt determinations of its specific heat at various temperatures for this purpose.

The only element which seemed to present the assemblage of characters required for the purpose contemplated was tellurium. I am indebted to Dr. T. K. Rose for a supply of tellurium which had already been refined by several dissolutions in aqua regia and precipitation by sulphurous acid. It was further purified by fusion with potassium cyanide, solution in water, and precipitation by exposure to air. Compounds of silver, nickel and tin with tellurium were made by the following process. Tellurium and the pure metals were weighed out in the proportions corresponding to the formulæ Ag_2Te, NiTe and $SnTe_2$ respectively, with a slight excess amounting to about 1 per cent. of tellurium. The materials were then fused together in a stream of hydrogen at a temperature sufficiently high to volatilise the excess of tellurium. The tellurides were obtained as black, crystalline, fusible substances, and were cast into cylindrical form by melting in a glass tube.

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* Regnault, 'Ann. Chim. Phys.' [3], 1, 129; Kopp, 'Phil. Trans.,' 1865.

† 'Phil. Trans.,' A vol. 201, p. 37 (1903).

The nickel used in the preparation of the telluride was in the form of soft wire drawn from metal obtained by electrolysis. For this I am indebted to the kindness of Dr. J. WILSON SWAN, F.R.S. Its specific heat was determined in the steam calorimeter in order to compare it with the fused nickel made for the previous experiments, but which was not found to be sufficiently ductile to admit of being drawn into wire.

SOFT Nickel Wire.

Range of temperature.	Specific heat.	Mean.
° C.		
22 to 100	·1087	·1086
20 ,, 100	·1082	
23 ,, 100	·1088	

The mean specific heat adopted as the result of the previous experiments on fused nickel was ·1084 for the same range of temperature.*

Alloys of silver and aluminium have also been examined. They were prepared by melting together the exact proportions of the pure metals. In the first the silver largely predominates, being in the ratio required by the formula Ag_3Al. The second contains aluminium in proportion corresponding to the formula $AgAl_{12}$, which represents 75·1 per cent. of aluminium and 24·9 per cent. of silver.

As in the results set forth in the previous paper, the specific heat adopted is the mean of several closely concordant experiments made at each range of temperature. The figures followed by E are estimated from the others which are the direct results of experiment. It will be seen (Table I.) that the value for specific heat increases with rise of temperature in every case except silver telluride, where the mean specific heats found between 15° and 309° and 390° C. respectively are less than at lower temperatures. This irregularity is attributed to the fact that during the later experiments the mass cracked and it was found necessary to re-melt it several times. This was done in hydrogen gas, and though no change in appearance was observed, some slight change in composition or structure may have been produced. These figures have, therefore, not been used in the subsequent calculations.

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* 'Phil. Trans.,' A, vol. 201, p. 38 (1903).

TABLE I.—Mean Specific Heats.

Range of temperature.	Tellurium.	Tin.	Silver-telluride. Ag_2Te.	Nickel-telluride. NiTe.	Tin-telluride. $SnTe_2$.	Silver-aluminium. Ag_3Al.	Aluminium-silver. $AgAl_{12}$.
° C.							
− 182 to 15	·0469	·0499	·0516	·0588	·0471	·0620	·1477
15 ,, 100	·0483	·0557	·0672	·0670	·0494	·0696	·1802
15 ,, 180	·0486 E	·0577	·0686	·0689	·0489	·0703	·1861
15 ,, 200	·0487	—	—	—	—	—	—
15 ,, 227	·0488 E	—	—	·0690 E	·0486	·0704	·1863 E
15 ,, 300	—	—	—	·0691	—	—	·1916
15 ,, 309	—	—	·0670	—	—	—	—
15 ,, 315	·0489	—	—	—	—	—	—
15 ,, 322	—	—	—	—	—	·0705	—
15 ,, 327	·0490 E	—	—	·0695 E	·0496	—	·1939 E
15 ,, 380	·0500	—	—	—	—	—	—
15 ,, 385	—	—	—	·0703	—	—	—
15 ,, 390	—	—	·0663	—	—	—	—
15 ,, 410	—	—	—	—	—	·0725	—
15 ,, 427	·0508 E	—	—	·0708 E	—	—	·2015 E
15 ,, 437	—	—	—	—	—	—	·2026
15 ,, 495	—	—	—	—	—	—	·2093

To these results may be added the mean specific heats of silver, nickel, and aluminium taken from the previous series of experiments.

TABLE II.—Mean Specific Heats.

Range of temperature.	Silver.	Nickel.	Aluminium.
° C.			
− 182 to + 15	·0519	·0838	·1677
15 ,, 100	·0558	·1084	·2100 E
15 ,, 180	·0561	·1101	·2189
15 ,, 227	·0565 E	·1120 E	·2208 E
15 ,, 327	·0577 E	·1175 E	·2247
15 ,, 427	·0581	·1233 E	·2356

The mean specific heats thus determined have been used, as in the former paper,[*] for the calculation of Q, the total heat measured in the calorimeter. The values of Q for the two elements tellurium and tin, the tellurides of nickel and tin, and the two alloys of aluminium and silver, have been plotted against absolute temperatures, and the results are shown in fig. 1, in which the curves are for the most part hyperbolic, those of tin, tellurium, and tin telluride approaching an elliptic form. In the case of tin, which melts at 232°, this is most probably due partly to incipient fusion at 180°, the highest experimental temperature, and is in accordance with experience.

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[*] Part II., 'Phil. Trans.,' A, vol. 201.

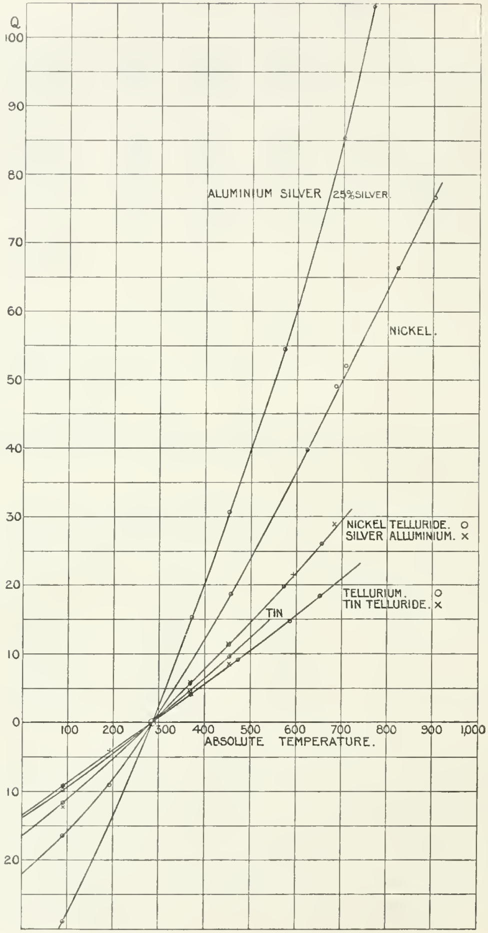
Q
100
90
80
70
60
50
40
30
20
10
0
10
20
100
200
300
400
500
600
700
800
900
1,000
ABSOLUTE TEMPERATURE.
ALUMINIUM SILVER 25%SILVER.
NICKEL.
NICKEL TELLURIDE. ○
SILVER ALLUMINIUM. ×
TIN
TELLURIUM. ○
TIN TELLURIDE. ×

Fig. 1.

Regnault, for example, found the molecular heats of easily fusible alloys to be very much higher than those of alloys of higher melting-point.*

The values of the true specific heats at successive temperatures on the absolute scale are given in the following table, and are exhibited graphically in figs. 2 and 3 :—

Table III.—True Specific Heats.

t abs.	Tellurium.	Tin.	Nickel-telluride. NiTe.	Tin-telluride. $SnTe_2$.	Silver-aluminium. Ag_3Al.	Aluminium-silver. $AgAl_{12}$.
° C.						
100	·0462	·0462	·0453	·0467	·0591	·1233
200	·0471	·0504	·0614	·0472	·0628	·1510
300	·0480	·0548	·0671	·0479	·0662	·1731
400	·0489	·0596	·0699	·0488	·0693	·1917
500	·0498	—	·0711	·0502	·0722	·2060
600	·0507	—	·0718	—	·0748	·2166
700	·0516	—	·0722	—	·0771	·2260
800	—	—	—	—	—	·2340

It is obvious that the curves for the specific heats of the compounds are of the same character as those for the metals aluminium, nickel, and silver given in the previous paper, and that the inclination of each is determined by the principal ingredient. Thus the curve for aluminium-silver containing 92 per cent. of silver is very near to the curve for that metal, while the curve for the alloy containing 75 per cent. of aluminium approaches the curve for pure aluminium.

The atomic heats of the elements are obtained by multiplying the specific heats by the respective atomic weights, which have been taken from the International Table.

Table IV.—Atomic Heats.

t abs.	Tellurium. Te = 126·6.	Tin. Sn = 118·1.	Silver. Ag = 107·12.	Nickel. Ni = 58·3.	Aluminium. Al = 26·9.
° C.					
100	5·85	5·46	5·00	3·35	3·30
200	5·96	5·95	5·65	5·12	4·66
300	6·08	6·47	5·98	6·14	5·52
400	6·19	7·04	6·13	6·81	6·06
500	6·30	—	6·22	7·19	6·41
600	6·42	—	6·29	7·43	6·65
700	6·53	—	6·32	7·58	6·81
800	—	—	—	7·70	—
900	—	—	—	7·80	

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* 'Ann. Chim. Phys.,' [3], 1, 137 and 183.

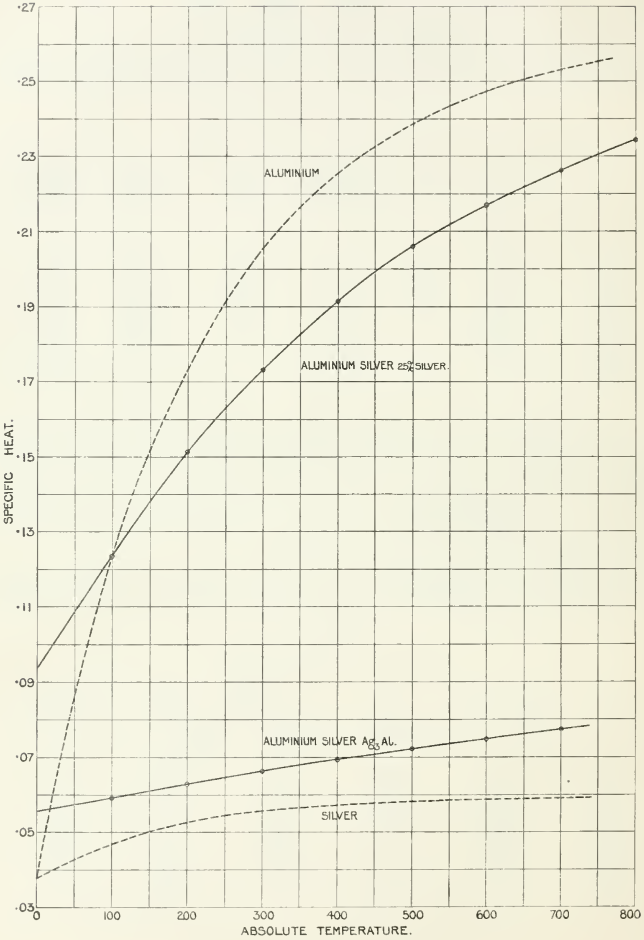
·27
·25
·23
·21
·19
·17
·15
·13
·11
·09
·07
·05
·03
SPECIFIC HEAT.
0
100
200
300
400
500
600
700
800
ABSOLUTE TEMPERATURE.
ALUMINIUM
ALUMINIUM SILVER 25% SILVER.
ALUMINIUM SILVER Ag_3Al.
SILVER

Fig. 2.

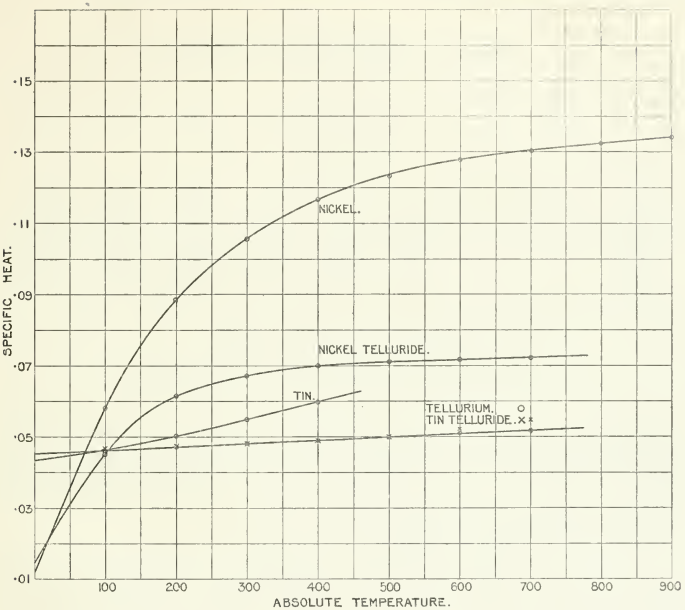

Fig. 3.

On the hypothesis that each atom in a compound behaves as it does in the solid element, the sum of the atomic heats of the elements entering into the compound should be equal to the molecular heat of the compound. The following table contains a comparison of the sum of the atomic heats, A, with the molecular heats, B, of the several compounds, that is, the product of the observed specific heat of the compound multiplied by the molecular weight in each case.

Table V.—Molecular Heats of Compounds.

t abs.	$SnTe_2$. A.	$SnTe_2$. B.	NiTe. A.	NiTe. B.	Ag_3Al. A.	Ag_3Al. B.	$AgAl_{12}$. A.	$AgAl_{12}$. B.
° C.								
100	17·16	17·33	9·20	8·38	18·30	20·58	44·60	53·01
200	17·87	17·51	11·08	11·35	21·61	22·38	61·57	64·92
300	18·63	17·77	12·22	12·41	23·46	23·06	72·12	74·42
400	19·42	18·10	13·00	12·92	24·45	24·14	78·85	82·41
500	—	—	13·49	13·15	25·07	25·15	83·14	88·56
600	—	—	13·85	13·28	25·52	26·05	86·09	93·12
700	—	—	14·11	13·35	25·77	26·85	88·04	97·16
800	—	—	—	—	—	—	—	—

A = calculated from the atomic heats of the elements.

B = calculated from the observed specific heats of the compounds.

The figures contained in this table show that in the cases of tin-telluride, nickel-telluride and the silver-aluminium alloy containing 92·28 per cent. of silver, there is a remarkably close approximation of the values under B to those under A, the differences between the two columns being throughout well within the limits of variation due to experimental error.

With regard to the aluminium-silver alloy containing only 24·9 per cent. of silver, however, there are differences which are somewhat greater. The values for silver-telluride must be regarded as open to suspicion, for reasons which have already been indicated, and they are not included in the table. If the *mean* atomic heats of silver-telluride are compared, it is found that the difference between the sum of the atomic heats and the molecular heat of the compound increases considerably with the temperature, as seen below:—

Temperature.	A.	B.	Difference.
° C.			
− 182 to + 15	17·06	17·59	0·53
15 „ 100	18·07	22·90	4·83
15 „ 180	18·17	23·38	5·21

This is perhaps due to some change taking place in the constitution of the solid. This, however, does not seem to be the explanation in the case of the aluminium-silver alloy, in which the differences between the two columns of figures, though not constant, do not increase appreciably—

B − A at 100° = 8·41,
200 = 3·35,
300 = 2·30,
400 = 3·56,
500 = 5·42,
600 = 7·03,
700 = 9·12.

Remembering what a large factor, 429·9, is used in calculating these figures, it will be seen that the differences are really small, being about 15 per cent. of the molecular heat at the lowest temperature and falling to about 3 per cent. at 300° abs. It will be noticed that in this case B is throughout larger than A.

The results of these experiments show that NEUMANN's law may be regarded as approximately valid for the specific heats at all temperatures. They also confirm the view that the specific heat of a solid is not a measure of the work done in separating the molecules of the substance, but that its amount is determined almost entirely by the nature of the atoms composing the physical molecules.

All the facts at our disposal show that there is not a great difference between the specific heats of elements in the solid and liquid states, but that in every case the latter is the greater, as shown in the following examples :—

TABLE VI.—Specific Heats.

Name.	Solid.	Liquid.	Authority.
Lead	·034	·040	PERSON.
Bromine	·084	·107	ANDREWS.
Gallium	·079	·080	BERTHELOT.
Phosphorus	·202	·204	PERSON.
Mercury	·032	·033	KOPP.
Bismuth	·030	·036	PERSON.
Tin	·058	·063	SPRING.

The *atomic* heat in the liquid state is thus in all cases greater than in the solid, and in the above cases ranges from 5·6 to 8·5. In the gaseous state at constant volume the atomic heat is, however, much smaller, being approximately for hydrogen 2·42, for oxygen 2·48, and for iodine, a solid at common temperatures and in some characters approaching the metals, 3·3. In respect to specific heat, therefore, the liquid state is not intermediate between the solid and the gaseous states. This may possibly be explained by the assumption that in the solid and in the gas at constant volume every molecule in the mass remains in the same condition relatively to every other molecule, for in the solid all are rigidly bound together by "cohesion," and in the gas all are equally free. In the liquid state there is reason to believe that there is a mixture of clusters or aggregates of molecules having different degrees of complexity, and that the effect of rise of temperature upon these is to cause dissociation of the more complex into simpler groups, a process which necessarily implies work done.

Notwithstanding the validity of NEUMANN'S law, the attempts which have been made to deduce the atomic heats of elements, such as oxygen, which do not admit of experiment in the solid state, cannot, however, be regarded as satisfactory. It is obvious that in such calculations whatever change in the molecular heat of the compound is induced by slight alteration of density, or of structure in the solid, is concentrated upon one element in the compound of any two, when it is assumed that the other enters into combination with the atomic heat it possesses in the elemental state. Taking the figures for the compounds containing silver, for example, the value deduced for the atomic heat of silver is found to vary considerably according to the nature of the compound selected. To calculate the atomic heat of silver from the mean molecular heat of the telluride—which is $\cdot0672 \times 340\cdot8 = 22\cdot90$ at the usual temperature of experiment, 0° to 100° C.—the value for tellurium is deducted and the remainder divided by 2. The result is 8·39.

Similarly the atomic heat of silver in the silver-aluminium, Ag_3Al, comes out as 6·19, and in the aluminium-silver, $AgAl_{12}$, as 9·67. The variations are still greater if a comparison is made over different ranges of temperature. Hence it appears probable that the values which have been calculated for hydrogen, oxygen, nitrogen and chlorine in the solid state are very far from the truth.

KOPP estimated the atomic heats of these elements in the solid state to be as follows:—

Hydrogen	2·3	Nitrogen	6·4
Oxygen	4·0	Chlorine	6·4

From its various compounds no approach to a uniform value for the atomic heat of carbon is to be found, but KOPP preferred the value 1·8, which is deduced from the specific heat of diamond.

Without here entering into a discussion of all these elements, it may be mentioned that in the case of hydrogen gas at constant volume the atomic heat is practically identical with that deduced from the specific heat of solid compounds, while that of oxygen is less, and that of carbon in the form of carbon dioxide gas is greater than the estimates thus made. JOLY found* the specific heat of air between 15° and 100° C. to be ·172, that of carbon dioxide gas ·173, and that of hydrogen 2·41, when under approximately equal pressures. The atomic heat of hydrogen gas is therefore 2·41. Assuming the specific heat of oxygen at constant volume very near to that of air, as it was shown to be many years ago, when at constant pressure, by REGNAULT, its atomic heat is about 2·7, which is a little greater than 2·48, the value deduced theoretically from REGNAULT's experiments at constant pressure. Lastly, taking ·173 for carbon dioxide and multiplying by 44, the value 7·61 is obtained as the molecular heat of carbon dioxide gas. If it be assumed that in the gaseous, as in the solid, state the atomic heat of each element is preserved in the compound, the atomic heat of gaseous carbon is left when the atomic heat of the oxygen in the dioxide is deducted. We thus obtain the value 2·65.

---

* 'Phil. Trans.,' A, vol. 182, p. 73 (1892).

This is greater than 1·8, the value chosen by KOPP, but falls between 2·89 and 2·42, the atomic heats of carbon, in the form of wood charcoal and natural graphite respectively, deduced from the experiments of REGNAULT between 0° and 100° C.

This deduction from data belonging wholly to the gaseous state is of interest because it is in accordance with the theoretical view that specific heat in a gas is not dependent on the temperature.

On the other hand, as the above experiments prove, atomic heats in the solid state, and probably also in the liquid state, are largely dependent on the temperature, the variation being abnormally great in solid carbon. It has also been shown that at the same temperature the atomic heats are widely different for different elements in the solid state; but notwithstanding this fact it has been proved that the molecular heat of a solid compound is approximately the sum of the atomic heats of its constituents at each temperature.

In conclusion, I desire again to express the obligations I am under to Mr. SIDNEY YOUNG and to Mr. LEONARD BAIRSTOW for their assistance.

Sun-spot Frequency

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VII. *An Enquiry into the Nature of the Relationship between Sun-spot Frequency and Terrestrial Magnetism.*

By C. Chree, *Sc.D., LL.D., F.R.S.*

Received February 8,—Read March 3, 1904.

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§ 1. In a recent paper,* termed here for brevity (A), I considered the relationship between certain phenomena of terrestrial magnetism on "quiet" days at Kew and Wolfer's sun-spot frequency. Some contemporaneous French and German magnetic data were also referred to, but in no great detail. I have now gone more fully into the question, in order to find out whether any of the phenomena conspicuous at Kew are peculiar to the place or to the period of time formerly dealt with.

---

* 'Phil. Trans.,' A, vol. 202, p. 335.

Assuming the relation between any magnetic quantity R—such as the daily range of declination—and sun-spot frequency S to be of the simplest type,

$$R = a + bS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

I determined the values of the constants a and b in a number of cases. The chief results were as follows:—

Supposing each month of the year treated separately, it was found that both a and b were conspicuously lower in "winter" (the four months November to February) than at the "equinoxes" or in "summer" (May to August); but b/a was larger in "winter" than at the "equinoxes," and larger at the "equinoxes" than in "summer." The b constants were generally fully larger at the "equinoxes" than in "summer," but to this there were exceptions.

The values of b/a were distinctly larger in the case of inclination and horizontal force than in the case of declination and vertical force; generally they were larger for inclination than for horizontal force, and larger for declination than for vertical force.

In all the elements b/a was larger when R represented the sum of the 24 hourly differences from the mean value for the day than when it represented the range in the diurnal inequality.

In (A) the letters D, I, H, V were used for declination, inclination, horizontal force and vertical force respectively. This practice is continued here when it tends to brevity.

§ 2. The first question now to be considered is as to the dependence of a and b on the particular period to which the magnetic and sun-spot data refer.

This is not so simple as might appear at first sight. Very few observatories have magnetic records extending over any large number of years, and in the few cases where such long records exist their homogeneousness is seldom, if ever, beyond dispute. There has usually been change both in the apparatus and its environment, and it is difficult even for those in immediate charge of an old observatory to know what allowance ought to be made to put old and new records on a common footing. This is especially true of V and I. The element where least uncertainty should prevail is D, but even here there is usually cause for doubt.

Milan Declination Ranges.

§ 3. RAJNA[*] has recently considered a long series of data representing the mean value for each year of the diurnal range of D at Milan. Since 1871 the range seems to have been determined from regular daily observations at 8 A.M. and 2 P.M.; but previous to that date there seems to have been some lack of strict uniformity. RAJNA has calculated values for a and b in the formula (1) from the data for the 59 years 1836 to 1894, and independently for the 24 years 1871 to 1894, employing the method of least squares. The formulæ he thus obtains, and one which he quotes as obtained by Wolfer, are as follows :—

---

[*] 'Rendiconti del R. Ist. Lomb.,' Series II., vol. 35, 1902.

$$\text{Rajna} \; . \; . \; . \; . \left\{\begin{array}{ll} 1836 \text{ to } 1894, & R = 5{\cdot}31 + 0{\cdot}047\,S \; . \; . \; . \; . \quad \text{(I.)}, \\ 1871 \text{ ,, } 1894, & R = 5{\cdot}39 + 0{\cdot}047\,S \; . \; . \; . \; . \quad \text{(II.)}, \end{array}\right.$$

$$\text{Wolfer} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; R = 5{\cdot}67 + 0{\cdot}040\,S \; . \; . \; . \; . \; \text{(III.)}.$$

Wolfer's value 0·040 for b is based on data from Christiania, Prague, Greenwich and Vienna, as well as from Milan. Rajna compares the values calculated from each of the three formulæ with the ranges observed at Milan from 1836 to 1901. Formula (I.) agrees rather better than (II.) with observation; formula (III.) seems distinctly inferior. The difference between the observed values and those calculated from (I.) varies from $-1'{\cdot}87$ in 1838 to $+1'{\cdot}87$ in 1866, the extreme values observed in R being $4'{\cdot}21$ and $12'{\cdot}03$. Since 1871 the agreement seems decidedly improved. Over the 24 years for which it was originally calculated (II.) gives a "probable error" of only $0'{\cdot}21$, the difference between observed and calculated values varying from $-0'{\cdot}49$ to $+0'{\cdot}53$.

Rajna himself notices that there are several long runs of the same sign in the differences between observed and calculated values. Thus from 1837 to 1850 the observed value is in excess of the calculated (from either (I.) or (II.)) 12 out of 14 times; on the other hand, in the 14 years 1854 to 1867, the calculated value is 13 times in excess. In the 11 years 1890 to 1900 (to which the Kew data treated in (A) referred) Rajna's calculated value from either (I.) or (II.) has been in excess 8 times, including every year since 1893.

The two first specified predominances of one sign are certainly in excess of what one would expect from pure chance. To throw some further light on the question, I have calculated values for a and b for the above-mentioned series of years. Instead of least squares, I grouped the years, following the method explained in (A), § 52. The grouping of years and the corresponding mean values of R and S were as follows :—

	Mean R.	Mean S.
Period 1837 to 1850—	9·270	68·9
Years of sun-spot maximum, 1837, 1838, 1839, 1847, 1848, 1849 . .	10·950	107·7
,, ,, minimum, 1841 to 1845	7·552	25·4
Period 1854 to 1867—	6·586	41·4
Years of sun-spot maximum, 1858 to 1862	8·176	76·1
,, ,, minimum, 1854, 1855, 1856, 1865, 1866, 1867 . .	5·245	14·3
Period 1890 to 1900—	7·163	41·7
Years of sun-spot maximum, 1892 to 1895	8·753	75·0
,, ,, minimum, 1890, 1899, 1900	5·587	9·5

The values found in the several cases for a, b and b/a, with the corresponding values from RAJNA's formula (I.), appear in Table I.

TABLE I.—Milan Declination Ranges.

Period of years . . .	1837 to 1850.	1854 to 1867.	1890 to 1900.	1836 to 1894 (RAJNA).
a	$6'\cdot 43$	$4'\cdot 62$	$5'\cdot 14$	$5'\cdot 31$
b	·0413	·0474	·0484	·047
$10^4 \times (b/a)$	64	103	94	89

§ 4. It is certainly satisfactory that the values of a and b for the period 1890 to 1900 differ so little from RAJNA's values for the long period 1836 to 1894. The probable error, employing my values of a and b, is only some 4 or 5 per cent. less than that found when employing RAJNA's values for the long period.

In considering the results for the two earlier periods, we must remember the want of homogeneousness referred to above. The mere existence of RAJNA's formula (I.) seems, however, evidence that, in his opinion, the want of homogeneousness is not serious, and the similarity of formulæ (I.) and (II.) to a certain extent supports this view.

The period 1837 to 1850 gives a very high value for a and a distinctly low value for b. The outstanding features of this period were the high mean sun-spot frequency, and the largeness of the declination range in the years of sun-spot minimum. Unless the results are very sensibly affected by heterogeneousness in the data, we must conclude that values calculated for a and b from a period as long as 14 years *may* depart somewhat widely from those calculated from a different equal or longer period. The range of variability would seem least in b and (naturally) greatest in b/a.

The value calculated for b from the period 1854 to 1867 agrees well with RAJNA's, but the value found for a is distinctly lower than his. The sun-spot frequency during this period presented similar features to those occurring in the 11 years 1890 to 1900.

Greenwich Declination and Horizontal Force Ranges.

§ 5. A second long series of data is that employed by Mr. ELLIS in two papers,* in which he compares D and H ranges at Greenwich with sun-spot frequency. Mr. ELLIS gives the observed D and H ranges from the diurnal inequalities for each month of the period 1841 to 1896. These are based on *all* days, excluding those of large disturbance. Corresponding "quiet" day data appear in Mr. Ellis' second paper for the 8 years 1889 to 1896. Mr. Ellis specifies several sources which may have introduced some heterogeneousness into the earlier data as compared to the later. Prior to 1848 there were only eye readings at 2-hour intervals, whereas subsequently hourly data were available. Prior to 1864, when the magnetographs were transferred to a new building, some uncertainty seems to have prevailed as to the temperature correction for H, and the data for 1864 itself seem to be interpolated. The data subsequent to 1864 would seem to be strictly homogeneous.

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* 'Phil. Trans.,' vol. 171, for 1880, p. 541; 'Proc. Roy. Soc.,' vol. 63, 1898, p. 64.

§ 6. Mr. Ellis employed no formula, and, whilst his graphical method appeals readily to the eye, it does not lend itself immediately to the present investigation.

I have accordingly calculated values for a and b for each month of the year in both D and H for the following periods: 1841 to 1896, 1865 to 1896, and 1889 to 1896 for both "all" and "quiet" days. In treating the first period, use was made of a group of sun-spot maximum years composed of 5 sub-groups each of 3 years, viz., 1847 to 1849, 1859 to 1861, 1870 to 1872, 1882 to 1884, and 1892 to 1894. The corresponding sun-spot minimum group consisted similarly of 15 years, made up of 5 sub-groups, viz., 1842 to 1844, 1854 to 1856, 1865 to 1867, 1877 to 1879 and 1888 to 1890.

For the period 1865 to 1896 the groups of sun-spot maximum and sun-spot minimum years were composed in either case of the last 3 sub-groups specified above.

For the period 1889 to 1896 the groups were: 1892 to 1895 for sun-spot maximum, and 1889 to 1890 for sun-spot minimum.

The values thus found for a, b and b/a appear in Tables II. and III. In addition to values for the individual months, the tables give values for winter, equinox and summer—each comprised of 4 months, as explained in § 1—and for the year. These seasonal and yearly values of a and b are simply arithmetic means of the individual monthly values; the seasonal and yearly values of b/a are derived from the seasonal and yearly values of a and b. The tables also supply corresponding data for Kew as given in (A), Table XL.

TABLE II.—Greenwich and Kew (unit 1′).

Ranges from Mean Monthly Diurnal Inequalities of Declination.

	a.					10^4b.					$10^4b/a$.				
	1841–96.	1865–96.	1889–96.		Kew.	1841–96.	1865–96.	1889–96.		Kew.	1841–96.	1865–96.	1889–96.		Kew.
			All.	Quiet.				All.	Quiet.				All.	Quiet.	
January . .	5·08	4·62	4·47	3·71	3·16	269	308	263	313	307	53	67	59	84	97
February . .	6·25	5·75	5·16	4·37	3·55	259	277	401	347	383	41	48	78	80	108
March . . .	7·32	7·10	6·86	6·29	7·13	502	504	602	719	526	68	71	88	114	74
April . . .	9·26	9·24	8·83	9·19	8·56	471	454	480	332	508	51	49	54	36	59
May . . .	8·57	8·45	8·24	8·22	9·34	444	443	487	544	401	52	52	59	66	43
June . . .	8·75	8·81	8·10	8·04	8·45	442	446	501	517	504	51	51	62	64	60
July . . .	8·70	8·83	7·94	8·08	8·75	421	428	542	488	387	48	49	68	60	44
August. . .	8·96	8·91	8·48	8·10	9·11	417	433	385	466	421	47	49	45	58	46
September. .	8·01	7·44	6·71	6·54	7·48	403	516	535	596	517	50	69	80	91	69
October . .	6·98	6·51	6·59	6·05	6·13	336	381	304	297	361	48	58	46	49	59
November. .	5·34	5·28	5·16	4·31	3·60	285	266	268	215	369	53	51	52	50	102
December . .	4·28	3·94	3·97	3·38	2·62	270	295	249	145	234	63	75	63	43	89
Winter. . .	5·24	4·90	4·69	3·94	3·23	271	287	295	255	323	52	59	63	65	100
Equinox . .	7·89	7·57	7·25	7·02	7·32	428	464	480	486	478	54	61	66	69	65
Summer . .	8·75	8·75	8·19	8·11	8·91	431	438	479	504	428	49	50	58	62	48
Year . . .	7·29	7·07	6·71	6·36	6·49	377	396	418	415	410	52	56	62	65	63

TABLE III.—Greenwich and Kew (unit 1γ).

Ranges from Mean Monthly Diurnal Inequalities of Horizontal Force.

	a.					10^3b.					$10^4b/a$.				
	1841–96.	1865–96.	1889–96.		Kew.	1841–96.	1865–96.	1889–96.		Kew.	1841–96.	1865–96.	1889–96.		Kew.
			All.	Quiet.				All.	Quiet.				All.	Quiet.	
January . .	14·2	10·7	11·4	11·2	11·6	157	147	116	181	118	110	137	102	162	101
February . .	13·8	11·0	13·0	15·1	10·5	203	187	143	219	172	147	170	110	145	164
March . . .	21·9	19·7	19·6	17·8	19·3	231	233	307	372	238	105	118	156	209	124
April . . .	33·3	32·2	29·7	31·5	25·7	265	246	310	262	284	79	76	104	83	111
May . . .	35·4	32·6	31·6	34·4	29·6	211	248	260	214	219	60	76	82	62	74
June . . .	36·7	33·4	34·1	37·2	32·6	223	301	281	194	142	61	90	82	52	44
July . . .	35·9	34·5	32·0	31·1	28·6	248	268	315	261	238	69	78	98	84	83
August. . .	35·8	32·9	33·3	38·3	31·6	169	234	238	154	160	47	71	72	40	51
September. .	32·8	30·3	31·7	30·1	27·0	155	205	178	223	192	47	68	56	74	71
October . .	27·1	24·5	24·1	29·0	22·2	142	182	201	130	171	52	74	83	45	77
November. .	15·9	13·7	13·8	15·3	13·0	168	180	177	237	225	106	132	128	155	173
December . .	13·6	8·2	10·6	9·1	6·7	108	145	91	114	131	80	176	86	125	193
Winter. . .	14·4	10·9	12·2	12·7	10·5	159	165	132	188	161	111	151	108	148	145
Equinox . .	28·8	26·6	26·3	27·1	23·5	198	216	249	246	221	69	81	95	91	94
Summer . .	35·9	33·4	32·7	35·2	30·6	213	263	274	206	190	59	79	83	58	62
Year . . .	26·4	23·6	23·7	25·0	21·5	190	215	218	213	191	72	91	92	85	89

§ 7. In the case of D, Table II., we see on the whole a close resemblance between the results from the two longest series of data. The later period shows, however, a slightly increased value of b and a slightly decreased value of a in winter, equinox, and the year as a whole. The last period, 1889–96, in the "all" day data shows a further increase in b and decrease in a; and the corresponding "quiet" day results give a still smaller value of a, especially in the winter months. The "all" and "quiet" day mean values of b for the year, from the 1889–96 period at Greenwich, are practically identical and very close to the corresponding Kew value. The values of b/a at Greenwich are in each season slightly larger for the "quiet" days than for the corresponding "all" days; and comparing the "all" day data amongst themselves we have an increase in b/a in passing from the longest to the mean period, and in passing from the mean period to the shortest period. This would imply that b/a has increased of late years.

At Greenwich, as at Kew, b is conspicuously lowest in "winter"; but no one of the four columns of Greenwich results gives so distinct an excess in the equinoctial over the summer value as appears at Kew, and, on the whole, we should infer that the equinoctial and summer values of b at Greenwich are practically equal.

At Greenwich, as at Kew, b/a is distinctly smaller in summer than in the other seasons; but the "winter" value of b/a at Greenwich, instead of markedly exceeding the values for the other seasons, as at Kew, would seem to be if anything slightly smaller than the equinoctial.

On the whole, the variation of b/a throughout the year at Greenwich is surprisingly small.

§ 8. The H ranges in Mr. ELLIS' tables are expressed in terms of the value of H at Greenwich. To make the results comparable with those for other stations, I have expressed the ranges in terms of 1γ as unit. In doing so, I treated H as constant for each period, and as possessing the following values:—

Period.	Value of H.
	C.G.S.
1841–96	·179
1865–96	·180
1889–96	·1829

Not knowing the exact procedure followed at Greenwich, I may not unlikely differ slightly from the exact values adopted there; but, for the purpose of the present enquiry, such small uncertainty as may exist is immaterial. The values of b/a are, of course, independent of the unit adopted.

The 1841–96 and 1865–96 data in Table III. present some conspicuous differences; a is larger in the former series than in the latter in every single month of the year, but b shows exactly the opposite phenomenon in 9 months out of the 12. This implies a very conspicuous difference between the phenomena in years prior and subsequent to 1864. Not improbably the want of homogeneousness in the earlier data, already referred to, may be partly accountable for the apparent change. At all events, the results for the final period, 1889–96, show no distinct progressive diminution of a or increase of b as compared to the period 1865–96. In fact. the "all" day data for these two periods, and the "quiet" day data for the shorter period, give almost identical values for the mean b for the year.

As regards the seasonal phenomena, each set of Greenwich data makes b conspicuously least, but b/a conspicuously largest, in "winter." The December value of b is invariably the smallest. The "all" and "quiet" day data for 1889–96, as already stated, give very nearly the same mean value of b for the year, but the "quiet" days' value for b is much the larger of the two in winter, and the smaller in summer. The "quiet" day data at Greenwich present a remarkable similarity to the corresponding Kew data, as is best seen by comparing the seasonal values for b/a. The fact, however, that the absolute values of both a and b are some 10 per cent. higher at Greenwich than at Kew is rather suggestive of some misapprehension as to the scale values at one or both observatories.

§ 9. In cases such as the present, a comparison of calculated and observed values is useful. The Greenwich data do not lend themselves very readily to this, as Mr. ELLIS does not give mean values for individual years, and the interest attaching to such a comparison for individual months of the year seems hardly sufficient to justify the necessary labour. In (A), § 75, some results were given for individual months of the year at Kew, but none for the year as a whole. This information is accordingly now supplied in Table IV., so far as concerns the ranges and the sum of the 24 hourly differences from the mean in the mean diurnal inequalities for the year in the several elements. The calculated values are derived from the values given for a and b in (A) Table XLIV. From the mathematical standpoint the nicety of agreement is best judged by considering the last line in Table IV., which shows what percentage the probable error in a calculated value is of the total variation exhibited by the element in the 11-year period. For practical purposes the mean difference between the calculated and observed values, and the percentage it forms of the absolute mean value of the element, are, however, fully as important.

It will be seen that the agreement is about equally good for the ranges and for the sum of the 24 differences. It is, on the whole, slightly better for D and H than for V and I.

TABLE IV.—Kew (Units 1′ for Angles, 1γ for Force Components). Observed less Calculated Values.

Year.	Range from mean diurnal inequality for the year.				Sum of the 24 hourly differences in the mean diurnal inequality for the year.			
	D.	H.	V.	I.	D.	H.	V.	I.
1890	+0·49	+ 0·5	—	—	+ 2·12	+ 0·5	—	—
1891	+ ·40	+ 3·0	+ 1·4	+0·21	+ 2·55	+ 16·5	+ 6·3	+0·86
1892	+ ·24	− 1·4	+ 1·5	− ·12	+ 2·22	− 6·8	− 0·2	− ·54
1893	+ ·32	+ 0·9	− 1·1	+ ·05	− ·86	+ 2·9	− 0·2	+ ·12
1894	− ·16	+ 0·8	− 0·9	+ ·09	− ·49	+ 9·9	− 3·7	+1·12
1895	− ·28	− 0·4	+ 0·4	− ·01	− 1·05	− 6·4	+ 0·8	− ·61
1896	− ·14	− 1·4	+ 0·4	− ·08	·00	− 6·6	+ 7·7	− ·76
1897	− ·53	− 1·1	− 1·8	− ·05	− 1·68	− 6·4	−11·4	− ·24
1898	− ·41	− 1·9	− 0·2	− ·13	− 1·11	− 14·0	+ 0·3	− ·78
1899	+ ·07	+ 0·3	+ 1·2	− ·01	+ ·25	+ 8·3	+ 4·8	+ ·50
1900	+ ·01	+ 0·6	− 0·9	+ ·07	− 1·93	+ 2·3	− 4·2	+ ·33
Mean difference calculated $\sim$ observed	0·28	1·12	0·98	0·082	1·30	7·3	4·0	0·59
Probable error	0·23	0·95	0·78	0·071	1·09	6·1	3·8	0·46
Mean value of element	7·90	26·2	18·0	1·43	41·6	154·5	96·9	8·30
Range of element	3·54	15·5	7·5	0·95	21·0	108·1	37·1	6·35
$\frac{\text{Mean difference} \times 100}{\text{mean value}}$	4	4	5	6	3	5	4	7
$\frac{\text{Probable error} \times 100}{\text{range of element}}$	6	6	10	7	5	6	10	7

Pawlowsk Data (59° 41′ N. lat., 30° 29′ E. long.).

§ 10. Magnetic observations at the three Russian magnetic observatories at Pawlowsk, Katharinenburg, and Irkutsk are published very nearly on parallel lines under the auspices of the Central Physical Observatory at St. Petersburg. The results are very complete, and are kept well up to date. For the present enquiry I have selected Pawlowsk and Katharinenburg. The former is, I believe, the furthest north magnetic observatory which has been in continued existence for any length of time. Its results include very full details of diurnal maxima and minima, and of the amplitudes of movements in magnetic storms. They are not confined to "all" days, but give in addition full particulars for selected "quiet" days, or "normal" days as they were called by WILD, to whom the idea of their separate treatment is due.

I have treated the Pawlowsk data for the 11 years 1890 to 1900 by the method of groups, taking the same combination of years as for Kew, viz., 1892 to 1895 for sun-spot maximum, and 1890, 1899, and 1900 for sun-spot minimum.

Table V. refers to the ranges from the mean diurnal inequalities for the several months of the year, both from "all" and "quiet" days. The seasonal and yearly values of a and b are arithmetic means from the included months, and these arithmetic means are employed in calculating the seasonal and yearly values of b/a.

TABLE V.—(Units 1′ for Angles, 1γ for Force Components.)

Ranges from Mean Monthly Diurnal Inequalities at Pawlowsk.

	Declination.						Horizontal force.						Vertical force.						Inclination.		
	a.		10^4b.		$10^4b/a$.		*a*.		10^3b.		$10^4b/a$.		*a*.		10^3b.		$10^4b/a$.		*a*.	10^4b.	$10^4b/a$.
	All.	Quiet.	All.	Quiet.	All.	Quiet.	All.	Quiet.	All.	Quiet.	All.	Quiet.	All.	Quiet.	All.	Quiet.	All.	Quiet.	All.	All.	All.
January	4·12	1·95	203	254	49	130	8·0	8·5	99	89	123	104	9·5	3·4	133	22	140	64	0·71	65	88
February .	4·45	2·30	399	482	90	209	8·8	10·2	254	169	288	165	6·3	4·2	493	25	777	58	0·75	114	152
March . .	6·12	6·12	693	735	113	120	22·8	19·5	247	316	108	162	11·9	8·8	484	16	407	18	1·51	156	103
April . .	8·92	9·30	482	477	54	51	33·4	31·2	314	258	94	83	15·3	10·5	206	60	135	57	1·95	202	104
May . .	9·58	9·97	601	676	63	68	35·3	32·0	271	311	77	97	18·0	11·8	190	38	105	32	2·04	158	78
June . .	9·60	10·91	587	522	61	48	32·3	34·2	373	230	115	67	9·5	9·3	238	66	252	71	1·95	205	105
July . .	9·55	9·92	507	532	53	54	31·0	30·9	393	247	127	80	5·7	10·3	381	62	674	60	1·90	211	111
August .	9·92	9·89	324	399	33	40	32·8	33·6	250	135	76	40	11·4	8·9	161	35	141	40	2·03	134	66
September	7·06	7·64	364	383	52	50	28·5	29·7	233	172	82	58	8·9	4·5	310	102	348	227	1·69	184	109
October .	4·81	5·49	495	340	103	62	22·5	21·9	224	213	100	97	7·8	4·9	287	60	365	122	1·47	170	116
November	3·78	2·87	502	307	133	107	10·1	10·1	189	212	187	210	4·4	3·8	395	22	894	57	0·65	156	95
December	3·82	1·85	201	202	52	109	7·9	4·6	74	143	94	308	7·4	3·2	171	22	230	67	0·65	60	92
Winter .	4·04	2·24	326	311	81	139	8·7	8·4	154	153	177	183	6·9	3·7	298	23	430	61	0·69	99	143
Equinox .	6·73	7·14	509	484	76	68	26·8	25·6	254	240	95	94	11·0	7·2	322	59	293	83	1·66	178	107
Summer .	9·66	10·17	505	532	52	52	32·9	32·7	322	231	98	71	11·1	10·1	242	50	218	50	1·98	177	90
Year . .	6·81	6·52	446	442	66	68	22·8	22·2	243	208	107	94	9·7	7·0	287	44	297	63	1·44	151	105

§ 11. Table V. presents several novel features. In the declination we see a conspicuous difference between the variation of a throughout the year on "all" and on "quiet" days. In December and January the "all" day value of a is more than double the "quiet" day value, and the excess of the "all" day value is also prominent in November and February. On the other hand, the "quiet" day value of a is, in general, distinctly the larger throughout the equinoctial and summer months. There is no such prominent difference between the "all" and "quiet" day values of b for declination. There are, of course, conspicuous differences in one or two individual months, but the seasonal and yearly values are closely alike. The rise of b/a in winter and its fall in summer are conspicuous, especially in the "quiet" days, where the phenomenon is even more prominent than at Kew. The mean values of b/a for the year from "all" and from "quiet" days are in close agreement with one another and with the corresponding Kew value.

In H the seasonal and yearly values of a in "all" and in "quiet" days are much alike. The winter and equinoctial values of b in the two cases are also nearly equal, but in summer the "quiet" day value is very decidedly the smaller. The mean "all" day value of b for the year is distinctly larger than the "quiet" day value, which is itself slightly in excess of the corresponding Kew value. The excess of b/a in winter is conspicuous in both "all" and "quiet" days; in the latter case the variation of b/a throughout the year is pretty similar to that at Kew.

In V the "all" and "quiet" day phenomena are vitally different. The fact that the mean diurnal range during the 11-year period for the "quiet" days was barely 40 per cent. of that for "all" days prepares one for a material difference between the phenomena in the two cases, but hardly for the "all" day mean yearly value of b being more than six times the corresponding "quiet" day value. The "quiet" day value of a for summer is not much less than the "all" day value, but in the equinoctial and winter months the latter greatly predominates. The value of b is greatest at the equinox in both "all" and "quiet" days; in fact, in "all" days the summer value of b falls short of the winter value, notwithstanding a marked depression in December and January. The variation of b/a throughout the year on "quiet" days is somewhat irregular. In "all" days we have the fall in summer and rise in winter seen at Kew, but there is a prominent depression in December and January. As regards the absolute magnitudes of b and b/a, the "quiet" day data are much closer than the "all" day to the Kew results.

Table V. gives only "all" day data for I, as no "quiet" day data for this element seem to be published.

Here again the phenomena resemble those observed at Kew, b being conspicuously small in winter and b/a small in summer. The values of a and b are, on the whole, distinctly larger than at Kew (where the diurnal range of I is less than at Pawlowsk), but the mean value of b/a for the year is 105×10^{-4} as compared with 111×10^{-4} at Kew.

§ 12. The data ascribed to the year in Table V. are arithmetic means from the 12 monthly values. Table VI. supplies values of a, b and b/a for the ranges from the mean diurnal inequality for the year as given in the Pawlowsk tables. They answer to the range data for Kew in Tables XLIII. and XLIV. of (A), and I give the results in the latter of these tables (calculated by least squares) for comparison.

Table VI.—(Units 1′ for angles, 1γ for Force Components.)

Mean Diurnal Inequality for the Year at Pawlowsk.

	Declination.			Horizontal force.			Vertical force.			Inclination.		
	a.	$10^4 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^4 \times b$.	$10^4 \times b/a$.
All days . .	5·74	400	70	20·7	211	102	8·1	265	326	1·24	126	101
Quiet days .	6·17	424	69	20·6	195	95	5·9	27	46	—	—	—
Kew . . .	6·10	433	71	18·1	194	107	14·3	81	56	0·87	125	145

§ 13. The Pawlowsk tables give for each month the mean of the differences between the daily maxima and minima, irrespective of their time of occurrence. The range thus obtained is, of course, larger than that from the mean diurnal inequality for the month, and is a quantity considerably more influenced by magnetic disturbances. The mean of the 12 monthly means may be regarded as the mean for the year of the absolute ranges in individual days. This is the quantity to which the results in the first line of Table VII. apply. The figures in the second line refer to the mean of the 12 monthly ranges, a monthly range being defined as the difference between the highest and lowest values recorded during the month. The third line in the table refers to the annual range, *i.e.*, the difference between the highest and lowest values recorded during the year. Owing to occasional losses of trace, monthly and annual ranges are sometimes under-estimated, especially at times of large disturbance. Both quantities are mainly dependent on the amplitude of disturbances; the mean monthly range is the better measure of the generally disturbed character of the year.

Table VII.—Pawlowsk (Units 1′ for angles, 1γ for H and V).

	Declination.			Horizontal force.			Vertical force.		
	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^2 \times b$.	$10^4 \times b/a$.	a.	$10^2 \times b$.	$10^4 \times b/a$.
Mean daily range . .	11·28	113	100	45·2	64	141	17·6	52	295
,, monthly range .	28·6	538	188	102	413	406	74	412	556
Annual range . . .	62·9	893	142	240	1226	510	338	698	207

§ 14. If we compare the results for the mean daily range in Table VII. with those for the range of the diurnal inequality in Table VI., we see that, though the values of a in Table VII. are about double those in Table VI. in the case of D and H, the increase in b is relatively so much greater that the values of b/a in Table VII. are some 40 per cent. in excess of those in the earlier table. This means that the amplitude of magnetic disturbance is in general more enhanced relatively in years of sun-spot maximum than is that of the regular diurnal inequality. The great rise in b/a as we pass from the mean daily to the mean monthly range in Table VII. is evidence of the same fact. That this rise in b/a is not continued, except in H, as we pass from the mean monthly to the annual range, is at least consistent with the view that the incidence of *exceptionally* large magnetic storms is determined by causes of which sun-spot frequency is no exact measure.

§ 15. Table VIII. supplies information as to the degree of accordance between the observed values of the ranges and those calculated from the values of a and b in Tables VI. and VII. Table VIII. is constructed on parallel lines to Table IV.

TABLE VIII.—Pawlowsk (Units 1′ for D and I, 1γ for H and V).

Observed less Calculated Values.

Year.	Ranges from mean diurnal inequalities.							Absolute mean daily ranges.		
	"Quiet" (normal) days.			"All" days.						
	D.	H.	V.	D.	H.	V.	I.	D.	H.	V.
1890	+0·2	+ 3	+ 1	+0·30	0	− 1	0·00	+ 0·06	− 1·1	− 0·2
1891	+0·3	+ 1	+ 2	+ ·15	+ 2	+ 2	+ ·10	+ 0·71	+ 2·5	+ 3·2
1892	+ ·6	− 4	+ 1	+ ·09	+ 1	+ 8	+ ·05	+ 1·51	+19·4	+17·5
1893	+ ·3	+ 2	+ 1	+ ·51	− 1	−11	− ·07	− 3·05	−20·5	−21·0
1894	+ ·3	+ 5	− 2	− ·28	+ 1	+ 1	− ·06	+ 0·33	+ 2·3	+ 3·4
1895	− ·8	− 1	− 1	− ·08	− 1	− 2	+ ·03	− 0·44	− 6·2	− 4·7
1896	− ·3	− 2	0	− ·02	− 1	+ 4	·00	+ 1·46	+ 2·0	+ 3·3
1897	− ·6	− 2	− 1	·00	0	0	+ ·02	+ 0·33	− 0·7	− 1·6
1898	− ·3	− 2	− 1	− ·55	0	+ 1	− ·02	+ 0·40	+ 5·0	+ 3·6
1899	− ·2	− 2	0	− ·20	+ 1	+ 1	+ ·04	+ 0·50	+ 4·9	+ 2·7
1900	+ ·3	+ 1	− 1	+ ·08	− 1	− 3	− ·08	− 1·81	− 7·6	− 6·2
Mean difference calculated ∽ observed .	0·38	2·3	1·0	0·21	0·8	3·1	0·043	0·96	6·56	6·13
Probable error . . .	0·30	1·8	0·8	0·19	0·7	3·2	·037	0·91	6·60	6·27
Mean value of element	7·94	28·7	7·0	7·41	29·5	19·2	1·77	16·0	71·8	39·3
Range of element . .	3·6	20	4	3·62	16	27	0·96	10·5	67·4	56·7
$\frac{\text{Mean difference} \times 100}{\text{mean value}}$	5	8	14	3	3	16	2	6	9	16
$\frac{\text{Probable error} \times 100}{\text{range of element}}$.	8	9	20	5	4	12	4	9	10	11

The calculated values of H and V employed in Table VIII. for the inequality ranges were taken only to the nearest 1γ, because the Pawlowsk tables go no nearer than this; but in the "all" days' D and I, both tables and calculation go to $0'{\cdot}01$. The agreement between calculated and observed values is much closer in D, H, and I than in V; and in D and H it is considerably closer for the "all" day than the "quiet" day results. Probably this only means that the fewness of the "quiet" days (sometimes only two or three a month) introduces an element of uncertainty which more than neutralises the effect of the greater regularity in these days.

§ 16. If the range of magnetic elements were largely dependent on influences which did not proceed *pari passu* with sun-spot frequency, then what we should expect to see in Table VIII. would be a notable occurrence of large + values in *all* the elements in some years, and of large − values in other years. The same result would follow if, while an intimate connection subsisted, it were not of the linear type assumed in (1).

So far as the inequality ranges in D, H, and I are concerned, there is no indication of such a phenomenon. There is indeed an excess of + signs from 1890 to 1894 and of − signs from 1895 to 1899, but the differences themselves are small, and those for the "all" and the "quiet" days show no kind of regular relationship. In the case, however, of the "all" day V inequality, and of the absolute daily ranges for all the elements, especially H and V, the observed values are conspicuously in excess of the calculated in 1892, and as conspicuously below them in 1893. This phenomenon seems due beyond a doubt to the influence of the disturbance element.

§ 17. With a view to further elucidation of the phenomenon described in the last paragraph, I have placed side by side in Table IX. data as to the mean value for each year of a variety of quantities which are affected in different degrees by magnetic disturbance. The small figures in brackets attached to the annual figures show the position which the year in question would occupy on a list which followed the order of magnitude of the quantity in question. If two yearly items are equal, a common number is attached. In the case of the years themselves, the attached figures indicate the order when the arrangement follows sun-spot frequency. It should, however, be noticed that the excess of sun-spot frequency in 1898 over 1897 was very trifling, and that the differences between 1899, 1900, and 1890 were not large.

In the case of the diurnal inequalities in D and H, quantities but little affected by disturbance, 1893 heads the list, just as it does in sun-spot frequency. In the case of the mean daily range—a quantity more influenced by disturbance—1892 and 1894 come to the front, and 1893 falls to the fourth place. Coming to the mean of the monthly ranges, we see 1892 and 1894 still more in advance, while in the case of H and V 1893 stands lower than 1898, a year of less than one-third its sun-spot frequency.

In the case of the annual range, 1893 has fallen to the eighth place in D and ninth place in V, whilst 1898 mounts to the third or even the second place.

In the mean daily, mean monthly, and annual ranges, 1892 and 1894 are as conspicuously in excess of what one would expect from sun-spot frequency as 1893 and 1895 are below it. Thus when we treat these four years as a unit, and compare it with a similar unit made up of the three years 1890, 1899, and 1900, we may arrive at a conspicuous connection between sun-spot frequency and amplitude of disturbance; but at the same time there is a marked absence of the close and regular connection in individual years which characterises the inequality ranges in D, H, and I.

TABLE IX.—Pawlowsk (Units 1′ for D, 1γ for H and V).

Year.	Diurnal inequality range.		Mean daily range.		Mean monthly range.			Annual range.		
	D.	H.	D.	H.	D.	H.	V.	D.	H.	V.
1890 (11) . . .	6·32 (8)	22 (10)	12·14 (10)	49 (10)	28·2 (11)	118 (11)	80 (11)	42·1 (11)	169 (11)	179 (11)
1891 (6) . . .	7·31 (6)	30 (5)	16·01 (6)	70 (6)	46·3 (7)	218 (7)	233 (5)	92·3 (6)	550 (5)	614 (4)
1892 (3) . . .	8·75 (2)	37 (3)	21·04 (1)	111 (1)	93·6 (1)	698 (1)	575 (1)	194·0 (1)	2416 (1)	1385 (1)
1893 (1) . . .	9·64 (1)	38 (1)	17·82 (4)	79 (4)	48·3 (4)	241 (4)	210 (7)	87·1 (8)	514 (6)	457 (9)
1894 (2) . . .	8·58 (3)	38 (1)	20·42 (2)	97 (2)	84·1 (2)	493 (2)	493 (2)	145·6 (2)	1227 (2)	878 (3)
1895 (4) . . .	8·22 (4)	33 (4)	18·07 (3)	80 (3)	47·4 (5)	220 (6)	223 (6)	73·9 (9)	395 (9)	534 (6)
1896 (5) . . .	7·39 (5)	29 (6)	17·46 (5)	74 (5)	52·4 (3)	232 (5)	236 (4)	88·7 (7)	574 (4)	608 (5)
1897 (8) . . .	6·79 (7)	26 (7)	14·57 (8)	61 (8)	43·8 (8)	201 (8)	170 (8)	101·1 (4)	449 (8)	480 (8)
1898 (7) . . .	6·25 (9)	26 (7)	14·70 (7)	67 (7)	46·6 (6)	276 (3)	242 (3)	118·9 (3)	1136 (3)	888 (2)
1899 (9) . . .	6·02 (11)	24 (9)	13·14 (9)	58 (9)	38·3 (9)	178 (9)	150 (9)	63·8 (10)	382 (10)	527 (7)
1900 (10) . . .	6·20 (10)	22 (10)	10·54 (11)	44 (11)	32·8 (10)	134 (10)	89 (10)	94·2 (5)	457 (7)	365 (10)
Means . .	7·41	30	15·99	72	51·1	274	246	100·2	752	629

§ 18. It was pointed out in (A), §§ 74 and 75, that whilst an intimate general connection between sun-spot frequency and diurnal magnetic ranges is unmistakable, it is open to doubt whether the mean values of these quantities for so short a period as a single month can be regarded as directly interconnected.

If both phenomena proceed from a common cause whose intensity of action at a given instant varies throughout the solar system, then it might possibly be better to compare monthly magnetic ranges with sun-spot frequency for a longer overlapping period.

As shown in (A), Table I., the mean sun-spot frequency for individual months of the year from the period 1890 to 1900 varied from 35·0 in November and 35·5 in March to 45·4 in June and August. Clearly, if the connection between sun-spot frequency and magnetic range is of the more general kind indicated above, the values we have found for b and b/a at Kew and Pawlowsk will be too large in months such as November and March and too small in months such as June and August.

To obtain an outside estimate of the uncertainty thus existing, I have calculated values for a, b and b/a for the "quiet" day Pawlowsk data, employing WOLFER'S smoothed sun-spot frequencies (Ausgeglichene Relativzahlen), each of which is a mean from observed values for 13 months, of which the individual month forms the central period. Table X. gives the mean seasonal and yearly values thus found; these answer precisely to the seasonal and yearly values based on observed sun-spot frequencies which appear in Table V.

TABLE X.—(Units 1′ for D, 1γ for H and V.)

Pawlowsk "Quiet" Days with WOLFER'S Smoothed Values (Ausgeglichene Relativzahlen).

	Declination.			Horizontal force.			Vertical force.		
	a.	$10^4 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.
Winter . . .	2·17	312	144	8·0	152	191	3·6	23	63
Equinox . .	7·29	428	59	26·2	216	82	7·4	55	74
Summer. . .	9·98	616	62	31·9	265	83	9·9	58	58
Year. . . .	6·48	452	70	22·0	211	96	7·0	45	65

So far as the mean yearly and winter values of a, b and b/a are concerned, Tables V. and X. are in practical agreement, but the equinoctial values of b in Table X. are decidedly lower, and the summer values decidedly higher, than the corresponding quantities in Table V. The fact that the equinoctial values of b/a for D and H in Table X. fall slightly below the summer ones seems hardly likely *à priori* to be a natural phenomenon, and it is not in accordance with the results obtained for Greenwich, in Tables II. and III., from the longer periods, where the variation of the mean sun-spot frequency from month to month is naturally less than in 1890 to 1900.

§ 19. The effect of the substitution of the smoothed sun-spot frequencies on the values of b and b/a from month to month is most easily followed by expressing the monthly values as percentages of their mean for the 12 months. Table XI. gives the mean of the results thus obtained for D and H, employing smoothed and observed sun-spot frequencies for the "quiet" days, and observed frequencies for the "all" days. The employment of smoothed frequencies for the "all" days would alter the results to about the same extent as it does in the "quiet" days.

The substitution of the smoothed for the observed sun-spot frequencies for "quiet" days removes an isolated prominence shown by the b variation in March, and removes slight depressions in June and August, but it produces a depression in September and adds materially to an already conspicuous prominence in May. Also the June depression and the March prominence are not apparent in the "all" day b variation using the observed sun-spots, and if we used smoothed frequencies for the "all" days we should have a marked depression in March and a largish prominence in June.

The b/a variation throughout the year proceeds most regularly when we use the observed frequencies.

TABLE XI.—Pawlowsk.

Variation of b and b/a throughout the Year.

	b.			b/a.		
	"Quiet" days.		"All" days.	"Quiet" days.		"All" days.
	Smoothed frequencies.	Observed frequencies.	Observed frequencies.	Smoothed frequencies.	Observed frequencies.	Observed frequencies.
January . . .	50	50	43	116	117	85
February. . .	96	95	97	188	187	180
March. . . .	116	159	128	89	135	124
April	123	116	119	66	63	76
May	171	151	123	90	78	75
June	126	114	142	59	55	90
July	124	120	137	63	63	89
August . . .	103	77	88	53	39	54
September . .	71	85	89	40	52	69
October . . .	84	90	101	67	75	113
November . .	72	86	95	121	147	170
December . .	64	57	38	248	188	75

Katharinenburg (56° 49′ N. lat., 60° 38′ E. long.).

§ 20. Table XII. gives values of a, b and b/a for the range of the diurnal inequality for each month of the year, and arithmetic means for the seasons and the year, at Katharinenburg, corresponding exactly to the "all" day results for Pawlowsk given in Table V.

In D. b appears decidedly less at Katharinenburg than at Pawlowsk, especially in winter. The dip in the December and January values of b in Table XII. is particularly striking. The summer and equinoctial values of b/a at Katharinenburg are very similar to those at Kew and Pawlowsk, but the winter value is much less, owing to the low values in December and January.

In H the mean b for the year is close to the Kew value, but the winter values of b and b/a are distinctly lower than at Kew and Pawlowsk.

In V, Katharinenburg occupies an intermediate position between the "all" and the "quiet" day results for Pawlowsk The mean value of b for the year in Table XII. is little over half the corresponding "all" day value at Pawlowsk, but it is more than thrice the "quiet" day value at Pawlowsk, and double that at Kew.

The winter values of b and b/a at Katharinenburg are exceptionally large, notwithstanding a marked depression in December and January.

TABLE XII.—Katharinenburg (Units 1′ for D and I, 1γ for H and V).

Ranges from Mean Monthly Diurnal Inequalities.

	Declination.			Horizontal force.			Vertical force.			Inclination.		
	a.	$b \times 10^4$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^4 \times b$.	$10^4 \times b/a$.
January .	3·09	61	20	7·0	91	130	6·1	95	156	0·55	61	112
February .	3·07	293	95	9·8	115	117	4·8	249	514	0·69	60	88
March. .	5·48	605	110	20·6	231	112	9·5	224	235	1·21	138	114
April . .	9·18	483	53	24·5	308	126	15·8	121	76	1·32	184	143
May . .	9·62	509	53	32·0	185	58	16·5	121	73	1·77	111	63
June . .	9·11	545	60	27·6	264	95	11·8	159	135	1·54	144	94
July . .	9·12	438	48	26·1	296	113	9·9	235	238	1·39	177	128
August .	9·04	310	34	27·5	197	72	11·5	102	89	1·56	114	73
September	6·54	351	54	23·7	201	85	7·8	137	175	1·41	130	92
October .	4·33	305	71	16·7	246	147	5·9	169	285	1·02	164	161
November	2·85	251	88	6·9	164	239	4·9	169	345	0·48	112	232
December.	2·68	114	42	7·7	38	50	5·4	96	177	0·54	38	71
Winter .	2·92	180	61	7·86	102	130	5·32	152	286	0·565	68	120
Equinox .	6·38	436	68	21·38	246	115	9·77	163	167	1·239	154	124
Summer .	9·22	451	49	28·30	235	83	12·44	154	124	1·563	137	87
Year . .	6·18	355	58	19·18	195	101	9·17	156	170	1·122	120	106

In I, both a and b are distinctly smaller at Katharinenburg than at Pawlowsk; in summer and the equinox they are very similar to the corresponding values at Kew. The winter value of b at Katharinenburg is decidedly less than at Kew, there being specially low values in December, January, and February. The mean value of b/a for the year is very close to the corresponding values at Kew and Pawlowsk.

§ 21. Table XIII. gives results for the mean of the absolute diurnal ranges for individual months, with corresponding seasonal and yearly values. The last mentioned correspond nearly to the Pawlowsk data in the first line of Table VII. (see § 22 for nature of difference).

In D the values of a in Table XIII., whilst invariably larger than the corresponding values in Table XII., are not very conspicuously so, except in winter. The values of b, however, in Table XIII., are conspicuously larger than those in Table XII., the mean values for the year being roughly one double the other. The difference between the values of b/a in the two tables, though less prominent, is unmistakable.

Table XIII.—Katharinenburg (Units 1′ in D, 1γ in H and V).

Monthly Means of Absolute Daily Ranges.

	Declination.			Horizontal force.			Vertical force.		
	a.	$10^4 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.
January . .	6·34	247	39	25·2	200	79	11·9	139	116
February . .	5·62	920	164	21·7	502	231	10·2	384	377
March . . .	8·44	974	115	31·7	468	148	15·6	383	245
April . . .	10·38	604	58	35·4	392	111	20·7	225	109
May	10·97	615	56	45·7	242	53	22·8	159	70
June . . .	10·00	666	67	36·1	397	110	17·9	215	120
July	9·61	706	73	32·6	538	165	15·0	324	215
August . . .	9·92	503	51	37·9	304	80	16·9	203	120
September . .	7·61	803	105	32·8	365	111	15·3	219	143
October . . .	6·68	661	99	28·9	365	126	11·4	255	223
November . .	4·67	910	195	18·7	440	236	8·6	313	365
December . .	5·15	427	83	21·5	194	90	10·1	155	152
Winter . . .	5·44	626	115	21·76	333	193	10·20	248	242
Equinox . .	8·28	761	92	32·21	398	123	15·76	270	172
Summer. . .	10·13	623	61	38·08	370	97	18·15	225	124
Year. . . .	7·95	670	84	30·68	367	120	14·70	248	168

In H the values of b in Table XIII. are on the average about double those in Table XII., but owing to the large values of a in Table XIII. the excess in its values for b/a is not striking, except in winter.

In V the values of a and b are again much larger in Table XIII. than in Table XII., but the seasonal and yearly values of b/a in the two are closely similar. In Table XIII. the December and January values of b/a are conspicuously low in all the elements as compared to the values for November and February.

§ 22. Table XIV. gives results for the range from the mean diurnal inequality for the year (corresponding to the Pawlowsk data in Table VI.), for the mean of the absolute daily ranges for the year (corresponding to the first line of Table VII.), for the mean of the 12 monthly ranges, and for the yearly range. The results for the second of these quantities, though practically accordant with those in the last line of Table XIII., are not absolutely identical. The figures in Table XIII. represent arithmetic means of a and b resulting from applications of formula (1) to individual months of the year. Table XIV. assumes the 12 monthly mean ranges for each year to be meaned, and these means dealt with by a single application of formula (1). The last two quantities dealt with in Table XIV. do not in reality accord very closely with the linear formula (1), but the figures at all events supply, as in the corresponding case at Pawlowsk, a rough measure of the amplitude of the fluctuation throughout a sun-spot period.

In all the elements included in Table XIV., as we pass from the range of the diurnal inequality to the mean absolute daily range, and thence to the mean monthly range—quantities increasingly influenced by magnetic storms—we see that whilst a increases, b increases in a greater ratio. so that b/a notably rises.

The fall of b/a as we pass from the mean monthly to the annual ranges in D and V may not improbably possess no real significance, but a similar phenomenon, it should be remembered, presented itself in the corresponding Pawlowsk results in Table VII.

Table XIV.—Katharinenburg (Units 1′ in D and I, 1γ in H and V).

	Declination.			Horizontal Force.			Vertical Force.			Inclination.		
	a.	$10^4 b$.	$10^4 b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^3 \times b$.	$10^4 \times b/a$.	a.	$10^4 \times b$.	$10^4 \times b/a$.
Mean diurnal inequality for the year	5·29	342	65	16·8	182	109	8·6	117	137	0·93	105	113
Mean of absolute daily ranges for the year	8·00	652	82	30·7	366	119	14·6	248	171	—	—	—
Mean of absolute monthly ranges for the year . .	18·56	2552	137	76·3	1680	220	46·3	1770	382	—	—	
Absolute yearly range	41·85	4750	113	146·3	4590	314	178·9	3600	201	—	—	—

§ 23. Table XV. shows the excess of observed over calculated values at Katharinenburg; it answers to Table VIII. for Pawlowsk.

The results for the diurnal inequalities in D, H, and I in Table XV. are similar to the corresponding "all" day results in Table VIII., but on the whole show a slightly less close agreement between theory and observation. In V, however, the agreement is distinctly better at Katharinenburg than at Pawlowsk.

In the case of the absolute daily range the agreement between observed and calculated values is particularly good in D, and it is closer in all the elements than at Pawlowsk. This may be ascribed to the fact that magnetic disturbances are larger at Pawlowsk than at Katharinenburg.

The difference between the observed and calculated values in the monthly range is somewhat large, and there is now clear indication that sun-spot frequency is not by itself a sufficient guide. The observed values in 1893 are conspicuously below, and those in 1892 and 1894 conspicuously above, the calculated. The deficiency in the observed values in 1895 and the excess in 1898 are also marked. Even in the absolute daily range in Table XV. there is a distinct depression in the observed values in 1893 and corresponding enhancement in 1892, though not to the same extent as in the corresponding case at Pawlowsk (see Table VIII.).

TABLE XV.—Katharinenburg (Units 1′ in D and I, 1γ in H and V).

Observed less Calculated Values.

Year.	Ranges from mean diurnal inequalities for the year.				Mean of absolute daily ranges.			Mean of 12 monthly ranges.		
	D.	H.	V.	I.	D.	H.	V.	D.	H.	V.
1890	+0·30	0	0	+0·04	+ 0·03	0	0	− 2·4	− 7	− 11
1891	+ ·35	+ 2	+ 1	+ ·07	+ ·45	+ 3	+ 2	− 1·3	− 9	− 6
1892	− ·04	+ 2	+ 2	+ ·02	+ ·62	+ 6	+ 4	+ 6·5	+ 54	+ 56
1893	+ ·64	− 1	− 2	− ·03	− ·94	− 8	− 8	−10·2	− 69	− 90
1894	− ·15	− 1	− 1	− ·03	− ·07	0	+ 1	+ 7·0	+ 54	+ 94
1895	− ·18	0	− 1	+ ·03	− ·19	− 1	− 1	− 6·7	− 44	− 60
1896	− ·21	+ 1	+ 2	+ ·01	+ ·39	+ 2	+ 2	+ 0·2	− 3	− 18
1897	− ·17	− 1	0	− ·05	− ·03	0	+ 1	+ 1·9	+ 3	− 6
1898	− ·44	− 1	− 1	− ·02	+ ·23	+ 1	+ 2	+ 5·0	+ 18	+ 28
1899	− ·37	+ 1	+ 1	+ ·06	+ ·04	+ 2	0	− 0·1	+ 11	+ 15
1900	+ ·27	− 1	− 2	− ·10	− ·51	− 4	− 3	− 0·1	− 8	− 3
Mean difference calculated ∼ observed	0·28	1·0	1·2	0·042	0·32	2·4	2·2	3·8	26	35
Probable error	0·23	0·8	1·0	0·035	0·30	2·5	2·2	3·5	25	34
Mean value of element	6·71	24·4	13·5	1·373	10·72	46	25	29·2	146	120
Range of element	3·50	15	11	0·87	5·27	33	23	27·5	180	230
Mean difference × 100 / mean value	4	4	9	3	3	5	9	13	18	29
Probable error × 100 / range of element	7	5	9	4	6	8	10	13	14	15

§ 24. Table XVI. supplies disturbance data at Katharinenburg, corresponding to those given in Table IX. for Pawlowsk. If we compare the 11-year means in the two tables we see convincing proof of the remark already made that Pawlowsk is more disturbed than Katharinenburg, the mean ranges in Table IX. being roughly double those in Table XVI.

The small bracketed figures in Table XVI. have the same significance as those in Table IX.

According to the mean monthly range—probably a better criterion than the annual range—1893 would seem to have been relatively less quiet at Katharinenburg than at Pawlowsk, but it stands much below 1892 and 1894. Whilst, however, all the Pawlowsk data made 1892 more disturbed than 1894, the monthly ranges at Katharinenburg give the first position to 1894. In all the columns 1890 appears as the least disturbed year. The monthly ranges—though not the annual ranges—assign to 1900 and 1899 the two next lowest places, the same positions as they occupy according to sun-spot frequency. But, as at Pawlowsk, 1895 is less disturbed, and 1898 much more disturbed than they should be if sun-spot frequency were the sole criterion.

TABLE XVI.—Katharinenburg (Units 1′ in D, 1γ in H and V).

Year.	Mean monthly range.			Annual range.		
	D.	H.	V.	D.	H.	V.
1890 . . .	18·0 (^{11})	82 (^{11})	48 (^{11})	29·3 (^{11})	163 (^{11})	117 (^{11})
1891 . . .	26·4 (^{8})	127 (^{7})	101 (^{6})	47·1 (^{8})	205 (^{10})	257 (^{5})
1892 . . .	43·8 (^{2})	253 (^{2})	232 (^{2})	116·9 (^{1})	837 (^{1})	591 (^{2})
1893 . . .	30·0 (^{4})	150 (^{3})	106 (^{4})	45·5 (^{9})	296 (^{5})	172 (^{10})
1894 . . .	45·5 (^{1})	261 (^{1})	278 (^{1})	90·0 (^{2})	674 (^{2})	849 (^{1})
1895 . . .	28·2 (^{6})	140 (^{5})	100 (^{7})	47·4 (^{7})	205 (^{9})	184 (^{9})
1896 . . .	29·4 (^{5})	143 (^{4})	102 (^{5})	48·6 (^{6})	320 (^{4})	244 (^{6})
1897 . . .	27·2 (^{7})	124 (^{8})	87 (^{8})	67·1 (^{4})	233 (^{7})	210 (^{7})
1898 . . .	30·4 (^{3})	139 (^{6})	122 (^{3})	83·8 (^{3})	338 (^{3})	471 (^{3})
1899 . . .	21·6 (^{9})	108 (^{9})	83 (^{9})	40·8 (^{10})	208 (^{8})	333 (^{4})
1900 . . .	20·9 (^{10})	84 (^{10})	60 (^{10})	61·6 (^{5})	237 (^{6})	191 (^{8})
Means . .	29·2	146	120	61·6	338	329

Batavia (6° 11′ S. lat., 106° 49′ E. long.).

§ 25. Prior to the introduction of electric tramways in 1899 the magnetic results at Batavia Observatory were treated with great completeness in the annual Batavia 'Magnetical and Meteorological Observations.'

Up to the end of 1900 the D and H results seem to have suffered comparatively little, but the V results even then were too disturbed for publication.

The Batavia magnetic records go back to 1882, but are incomplete until 1884. Inspection of the vertical-force data for 1884 and 1885 created some misgivings, which gathered force from an editorial statement that the scale value in that element was at first very variable and remained so until a new magnet was introduced.

After considering all the circumstances, I decided to confine myself to the results for the 12 years 1887 to 1898, coming down to the latest time at which all the elements were free from electric-tram effects. This period has the advantage of supplying a sun-spot minimum group of years 1887 to 1890 equal in length to the sun-spot maximum group 1892 to 1895.

The Batavia tables give not merely the hourly values, but also the sum of the 24 differences from the mean, in the monthly diurnal inequalities.

In (A) I found the sum of the 24 differences in D and H to show the sun-spot connection even more prominently than the ranges. Accordingly I decided to use the sum of the 24 differences at Batavia, in preference to the ranges, when dealing with the diurnal inequalities for the individual months, and to employ the sum of the 24 differences as well as the ranges when dealing with the mean diurnal inequalities for the year.

§ 26. Before giving the results, I would draw attention to a feature wherein Batavia differs widely from European stations. At Kew, for instance, D, H, V, and I all show a large variation in the amplitude of the diurnal inequality throughout the year. The range is three or four times as large at midsummer as at midwinter, and the way in which the range, or the sum of the 24 differences, varies throughout the year is pretty similar for all the elements. Thus, assuming that the mean diurnal inequality for the year were derivable from a potential, one could obtain a fair first approximation to the mean diurnal inequalities for individual months by multiplying this potential by appropriate numerical factors.

How exceedingly far this is from being the case at Batavia will be seen on inspection of Table XVII. This gives the sum of the 24 differences in the diurnal inequalities for each month of the year, with their mean, and the sum of the 24 differences in the mean diurnal inequality for the year. Batavia being in the Southern hemisphere, May to August are the "winter" months.

The D data in Table XVII. proceed, on the whole, like European data. In V, too, the lowest value occurs in the winter months, but there is likewise a low value in December. While the average value for I from the four winter months is below the mean for the year, the lowest values of all occur in November and December. In H, three out of the four winter months show values above the average, while the four summer months are all below the average. Thus no two elements behave alike, and the phenomena exhibited by H are more nearly opposite than parallel to those observed in high latitudes.

TABLE XVII.—Batavia, 1887–1898 (Unit $1'$ in D and I, 1γ in H and V).
Sum of the 24 Hourly Differences in the Mean Diurnal Inequality for the Month.

	Declination.	Horizontal force.	Vertical force.	Inclination.
January	22·04	313·8	276·7	31·77
February	22·55	310·4	289·7	32·36
March	16·87	357·0	282·1	33·72
April	11·96	373·0	231·3	30·87
May	12·34	346·3	189·5	26·85
June	10·26	322·5	189·8	26·02
July	12·14	347·1	202·6	27·84
August	15·25	373·7	187·4	27·87
September	18·56	396·2	214·7	30·08
October	20·57	348·5	249·8	30·51
November	22·06	281·9	215·8	25·82
December	22·01	263·4	196·7	23·82
Arithmetic mean from 12 months	17·22	336·2	227·2	28·96
From mean diurnal inequality for the year	13·73	334·5	216·6	28·36

§ 27. The results pointed out in the last paragraph help to explain some novel features in Table XVIII., which gives the values obtained for *a*, *b* and *b/a* by applying the method of groups to the sums of the 24 differences in the diurnal inequalities for the several months. As in similar tables, the seasonal and yearly values of *a* and *b* represent arithmetic means for the included months.

In D and V the lowest values of *a* are found in winter, but in H and I the lowest values occur in November and December, that is, at midsummer.

In D and V, again, *b* is distinctly below the average in winter, though not nearly to the same extent as in Europe, and the winter value for I is less than the summer value; but in H the winter value exceeds the summer.

In D, *b/a* is distinctly smaller in summer than in the other seasons; but in I, H and V the summer value is a trifle the highest. The winter and equinoctial values of *b/a* are almost identical in all the elements. As compared to northern stations, the variation in *b/a* throughout the year is exceedingly small.

TABLE XVIII.—Batavia. Sum of 24 Hourly Differences, 1887–98 (Units 1′ in D and I, 1γ in H and V).

	Declination.			Inclination.			Horizontal force.			Vertical force.		
	a.	$10^3 \times b$.	$10^4 \times b/a$.	*a*.	$10^3 \times b$.	$10^4 \times b/a$.	*a*.	$10^2 \times b$.	$10^4 \times b/a$.	*a*.	$10^2 \times b$.	$10^4 \times b/a$.
January .	18·09	109	60	25·37	178	70	244·7	192	78	225·4	142	63
February	19·11	87	45	25·70	168	66	232·4	197	85	238·8	128	54
March .	13·52	105	78	26·65	222	83	266·8	283	106	232·6	155	67
April . .	9·94	55	55	25·74	139	54	309·6	171	55	196·0	96	49
May . .	9·60	68	71	20·00	171	85	264·2	204	77	143·0	116	81
June . .	8·62	40	46	19·87	149	75	241·1	197	82	151·5	92	61
July . .	9·24	69	74	22·29	131	59	259·2	208	80	174·8	66	38
August .	12·16	69	57	22·27	125	56	281·3	207	74	156·4	69	44
September	14·04	108	77	22·51	181	80	303·9	221	73	161·4	127	79
October .	17·87	73	41	24·73	156	63	265·7	224	84	215·6	93	43
November	19·52	78	40	19·20	203	106	212·3	214	101	165·7	154	93
December	18·32	95	52	19·12	121	63	207·5	144	70	163·2	86	53
Winter .	9·90	61	62	21·11	144	68	261·5	204	78	156·4	86	55
Equinox .	13·84	85	61	24·91	174	70	286·5	225	78	201·4	118	58
Summer .	18·76	92	49	22·35	167	75	224·2	187	83	198·3	128	64
Year . .	14·17	80	56	22·79	162	71	257·4	205	80	185·4	110	60

§ 28. In applying the method of groups, it is evidently desirable that one group of years should fall near the middle of the period dealt with, and that part of the second group should precede, and part follow it. This arrangement helps to eliminate any long-period variation, or any gradual change in the conditions. The period 1887 to 1898 being by no means ideal in the above respect, in dealing with the diurnal inequality for the year, I have employed both the method of least squares and the method of groups. If large differences had presented themselves between the results from the two methods, it would have become necessary to reconsider Table XVIII. As the question of the reliance to be placed on the method of groups is important, I give the results from both methods in Table XIX. The agreement, it will be seen, is least good in D, but it will, I think, be allowed that even there it leaves little to be desired.

TABLE XIX.—Batavia (Units 1′ in D and I, 1γ in Forces).

Mean Diurnal Inequalities for the Year.

		Declination.		Inclination.		Horizontal force.		Vertical force.		Total force.	
		Groups.	Least squares.	Groups.	Least squares.	Groups.	Least squares.	Groups.	Least squares.	Groups.	Least squares.
Ranges.	a	2·455	2·470	3·61	3·60	38·74	38·74	30·13	30·11	20·94	20·90
	$10^4 \times b$	183	179	215	218	2738	2739	1550	1559	1541	1552
	$10^5 \times b/a$	746	725	597	605	707	707	514	518	736	743
Sum of 24 differences	a	10·30	10·34	22·21	22·19	258·5	258·0	173·3	173·1	145·9	145·4
	$10^3 \times b$	88·8	87·6	159·1	159·8	1967	1980	1121	1127	1190	1204
	$10^5 \times b/a$	862	847	716	720	761	767	647	651	816	828

§ 29. In the case of I, H and V the values of a, b and b/a given in Table XIX. for the sum of the 24 differences in the mean diurnal inequality for the year do not differ much from the yearly mean in Table XVIII. In the case of D, however, the results in the two tables are widely different, the a in Table XVIII. being nearly 40 per cent. in excess of that in Table XIX. The closeness of the values of a in the two tables in I, H and V, and their divergence in the case of D, is what we might anticipate from the figures for the sums of the 24 differences in the last two lines of Table XVII. The real inference to be drawn is that in D the hours of maximum and minimum vary somewhat widely from month to month, though apparently to a smaller extent in years of sun-spot maximum than in years of sun-spot minimum.

The data in Table XIX. for Batavia correspond exactly to those given for Kew in (A), Table XLIV.

In D the Batavia value of b for the 24 differences is almost exactly the third of the Kew value; in the case of the ranges the Batavia value is relatively larger, but still less than half that at Kew, Greenwich or Pawlowsk.

The D values of b/a, however, at Batavia and Kew are nearly equal.

In I the Batavia value of b is 70 per cent. larger than the Kew in the case of the ranges, and almost exactly double in the case of the 24 differences. The Batavia values of a, however, so much exceed the Kew that b/a is more than twice as big at Kew as it is at Batavia.

In H the Batavia values of b are both roughly 50 per cent. in excess of the Kew, but the Kew values of b/a are more than 50 per cent. in excess of those at Batavia.

In V the values of b are again very much larger at Batavia than at Kew; in the 24 differences the Batavia value of b/a is somewhat the higher, but in the ranges it is slightly the lower.

§ 30. The relation between the values found for b/a is probably the best measure of the relative importance of the sun-spot connection in any two cases. Applying this criterion to the 24 differences and range results obtained by least squares in Tables XIX., we obtain the following values for the ratio of

(b/a) from sum of 24 differences : (b/a) from ranges :—

Declination.	Inclination.	Horizontal force.	Vertical force.	Total force.
1·168	1·191	1·085	1·257	1·115

The mean of the first four of these ratios is 1·18 and the corresponding figure for Kew (as deducible from (A), Table XLIV.) is 1·19. Thus the greater variability of the sum of the 24 differences with sun-spot frequency observed at Kew is also seen at Batavia, and to approximately the same extent.

§ 31. The Batavia publications record the values of the constants in the 24-hour and 12-hour terms of the Fourier series

$$c_1 \sin (t + \alpha_1) + c_2 \sin (2t + \alpha_2) +$$

for the mean diurnal inequality for the year. Here c_1, c_2 replace the Batavia notation A_1, A_2.

Table XX. gives the values which I have found for a, b and b/a in this case from the method of groups. The results should be compared with those given for Kew and Wilhelmshaven in (A), Table XLII., though the slight difference in the method of obtaining the Kew results should be noted.

In I, H and V the values of b/a in Table XX. are nearly alike in c_1 and c_2, and they approach fairly closely to the corresponding values in Table XIX. applicable to the ranges of the mean diurnal inequalities. In declination and total force, however, the values of b/a in Table XX. are decidedly higher for c_1 than for c_2. This phenomenon was observed at both Kew and Wilhelmshaven in the case of the declination and the westerly component.

TABLE XX.—Batavia, 1887–98 (Units 1′ for Angles, 1γ for Forces).

Amplitudes of 24-hour and 12-hour Terms in Fourier Series for Mean Diurnal Inequality for the Year.

	Declination.		Inclination.		Horizontal force.		Vertical force.		Total force.	
	c_1.	c_2.	c_1.	c_2.	c_1.	c_2.	c_1.	c_2.	c_1.	c_2.
Mean value of amplitude (for 12 years) . .	0·748	0·778	1·793	0·837	20·89	9·07	13·62	7·27	11·71	5·45
a	0·548	0·614	1·427	·663	16·13	7·13	11·14	5·90	8·84	4·43
$10^4 \times b$ . . .	52	42	95	45	1233	502	641	353	744	263
$10^4 \times b/a$ . . .	94	69	66	68	76	70	58	60	84	60

§ 32. Disturbances have special attention paid them at Batavia. Following a practice, of which SABINE was an advocate, a reading at Batavia is regarded as disturbed when its difference from the mean reading at that hour during the month reaches or exceeds a certain limit. The limiting values adopted at Batavia are 1′·3 in D and 11γ in H and V.

The arbitrary nature of such criteria, and the difficulty of justifying one limiting value in preference to another, have been more than once pointed out. It is arguable that the limit should vary with the season of the year, and even with the sun-spot frequency. In a European station, for instance, the range of the regular diurnal inequality near sun-spot maximum at midsummer is very large compared to that near sun-spot minimum at midwinter, and a good deal might be said for a limiting value which bore a fixed ratio to the range from the mean diurnal inequality for the month.

The disturbed values which exceed the hourly mean, and those which fall below it, are termed respectively "positive" and "negative" disturbances; they are in the first instance treated separately at Batavia, tables being given of the sum of the values of the disturbances and of their number. A final summary gives the aggregates of the positive and negative totals treated numerically. Table XXI. gives these aggregate values and numbers as published in the annual Batavia 'Observations.'

The number of disturbances in D is less than half that in V, and little over a quarter that in H. We cannot, however, draw any safe inference as to one element being absolutely more or less disturbed than another. If we calculate the ratios borne by the disturbance limits accepted at Batavia to the mean ranges of the diurnal inequalities for the year in the respective elements, for the period 1887 to 1898, we find the following results:—

	D.	H.	V.
Disturbance limit / range =	0·41	0·22	0·30

If instead of the ranges from the mean diurnal inequality for the year we had taken the arithmetic mean of the 12 monthly inequality ranges, we should have obtained a somewhat smaller fraction in the case of D. But the figures are at least suggestive that the explanation of the great difference in the number of disturbances in D, H, and V may be largely due to the disturbance limit being less exacting in one element than another.

TABLE XXI.—(Units for "Values" 1′ in D, 1γ in H and V.)

Aggregate Values and Numbers of Disturbances at Batavia.

Year.	Declination.		Horizontal force.		Vertical force.	
	Values.	Numbers.	Values.	Numbers.	Values.	Numbers.
1887	339·3	210	17,160	1023	9,612	671
1888	237·3	149	16,339	933	12,709	807
1889	237·5	153	11,686	700	11,581	783
1890	354·0	252	6,227	346	4,781	301
1891	425·4	262	22,605	1208	16,394	1016
1892	1020·8	571	40,582	1786	20,295	1095
1893	730·8	427	23,731	1286	11,021	715
1894	840·2	462	37,239	1666	13,418	751
1895	616·0	360	23,595	1380	11,441	757
1896	458·2	286	19,983	1139	5,790	409
1897	464·5	286	14,187	815	5,995	408
1898	434·2	262	18,605	1030	8,485	548
Means	513·2	307	20,995	1109	10,960	688

§ 33. On examining Table XXI. it will be seen that the number and aggregate value, though generally increasing or decreasing together, are far from being in a constant ratio in any of the elements. As to which is the better measure of disturbance, opinions may well differ; but the aggregate values constitute probably the nearest parallel to the Pawlowsk and Katharinenburg data in Tables IX. and XVI.

According to both numbers and aggregate values, 1893 was less disturbed than 1892 or 1894, but its relative quietness is not so conspicuous, especially in D and V, as it was at Pawlowsk or even Katharinenburg.

Table XXI. must, of course, receive contributions—at least, in the case of H and V—from a number of days which are not days of large disturbance; but if this were the true explanation, we should expect the position of 1893 at Batavia and Pawlowsk to differ more in the case of H than in that of D, which is the reverse of what happens.

§ 34. Table XXII. gives values of a, b and b/a calculated for the data in Table XXI. The value of b/a answering to the "aggregate value" is in each case greater than that answering to the "number"; and, except in the case of V, both values of b/a are considerably higher than the corresponding yearly values in Tables XVIII. and XIX.

If we compare Table XXII. with Table XIV. for Katharinenburg, we see that in V the Batavia disturbance values of b/a are much less than the lowest value of b/a at Katharinenburg, viz., that for the diurnal inequality. In H the Batavia disturbance values of b/a are similar to the value of b/a in the absolutely monthly range at Katharinenburg. In D, however, the Batavia disturbance values of b/a are much in excess of any corresponding value at Katharinenburg.

The way in which disturbance influences the records at the two places are thus widely different.

Table XXII.—Batavia "Disturbances," 1887–98 (Units for "Values" 1′ for D, 1γ for H and V).

	Declination.		Horizontal force.		Vertical force.	
	Aggregate values.	Numbers.	Aggregate values.	Numbers.	Aggregate values.	Numbers.
a	217·7	153·7	10,312	657·9	8425	578·9
b	7·65	3·96	277	11·7	65·6	2·84
$10^4 \times b/a$ . .	351	258	268	178	78	49

§ 35. Table XXIII. compares observed and calculated values in the mean diurnal inequality for the year at Batavia, and in the aggregate value of the disturbances. The values employed for a and b in the case of the ranges are those calculated by least squares.

The nicety of agreement in the case of the ranges is very similar to what has been already observed at Kew, Pawlowsk and Katharinenburg; and, as at Kew, the agreement is practically the same for the 24 differences as for the ranges. As has been generally observed elsewhere, the agreement is least good in the case of V.

In the case of the aggregate value of the disturbances, the agreement is pretty similar to what was found for the mean of the absolute monthly ranges at Katharinenburg; and, as elsewhere, the failure of the formula to account satisfactorily for the phenomena observed in 1892 and 1893 is conspicuous. Unlike Pawlowsk and Katharinenburg, Batavia shows, however, no abnormal excess of disturbances in 1898.

TABLE XXIII.—Batavia (Units 1′ for D, 1γ for H and V).

Observed less Calculated Values.

Year.	Mean diurnal inequality for the year.						Aggregate values of disturbances.		
	Ranges.			24 differences.					
	D.	H.	V.	D.	H.	V.	D.	H.	V.
1887	−0·25	+ 0·5	− 2·7	− 0·77	+ 2·2	− 18·6	+ 21·4	+ 3,235	+ 327
1888	− ·07	+ 0·9	+ 0·9	− ·28	+ 5·0	+ 0·2	− 32·4	+ 4,146	+ 3,838
1889	+ ·16	− 0·3	+ 3·0	+ ·36	− 1·1	+ 13·7	− 28·4	− 368	+ 2,743
1890	+ ·09	+ 0·4	+ 1·3	+ ·21	+ 8·4	+ 18·0	+ 82·0	− 6,049	− 4,110
1891	− ·22	+ 2·4	+ 5·3	− ·88	+ 15·2	+ 36·9	− 64·6	+ 2,448	+ 5,633
1892	+ ·16	+ 0·6	+ 2·8	+ ·40	+ 12·3	+ 17·9	+244·6	+10,082	+ 7,080
1893	− ·02	+ 1·1	+ 0·6	·00	+ 8·9	+ 3·9	−136·4	−10,061	− 2,975
1894	− ·04	− 1·6	− 1·8	− ·03	− 9·8	− 9·3	+ 25·7	+ 5,355	− 125
1895	− ·08	+ 1·4	+ 0·6	− ·53	− 0·3	− 0·7	− 91·4	− 4,418	− 1,184
1896	+ ·03	− 3·8	− 5·1	+ ·41	− 29·1	− 34·2	− 79·3	− 1,890	− 5,378
1897	+ ·19	− 0·6	− 3·4	+ ·66	− 6·3	− 23·0	+ 46·3	− 3,371	− 4,149
1898	+ ·04	− 1·0	− 1·6	+ ·44	− 5·5	− 5·1	+ 12·2	+ 909	− 1,692
Mean difference calculated ∽ observed	0·112	1·22	2·42	0·414	8·7	15·1	72·1	4,361	3,270
Probable error . .	0·096	1·09	2·02	0·344	8·1	13·4	61·2	3,734	2,730
Mean value of element	3·16	49·3	36·1	13·73	334·5	216·6	513	20,995	10,960
Range of element .	1·52	22·9	14·9	7·12	165·6	104·0	783	34,355	15,514
$\frac{\text{Mean difference} \times 100}{\text{mean value}}$	4	2	7	3	3	7	14	21	30
$\frac{\text{Probable error} \times 100}{\text{range of element}}$	6	5	14	5	5	13	8	11	18

Mauritius (Lat. 20° 6′ S., Long. 57° 33′ E.).

§ 36. Owing to the novelty in some of the features of the Batavia results, examination of the data from a second tropical station seemed desirable. I have accordingly made use of a number of tables of magnetic results* at Mauritius, published in a convenient form in 1899. In D, data are given for the period 1875 to 1890, in H for 1883 to 1890, and in V for 1884 to 1890. The shortness of the two latter periods, and the fact that the data are not contemporaneous with those for most of the other stations, are drawbacks, but there is small choice of magnetic data in low latitudes.

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* 'Mauritius Magnetical Reductions,' edited by T. F. CLAXTON, F.R.A.S., Director Royal Alfred Observatory, Mauritius, 1899.

On examining the tables, I found that the mean D ranges in 1881 and 1882—years of fairly large sun-spot frequency—showed a remarkable depression, being only about half those in 1880 and 1883, and in the preface I found the following editorial reference to some readjustment of the declination magnetograph in December, 1882 : "In the latter part of the year 1882 the effect of torsion on the magnetograph is very pronounced." As the phenomenal smallness of the range seems to have ceased with the readjustment, and as the Milan and Greenwich records show no parallel to the reduction of the ranges in 1881 and 1882, I have omitted these years entirely from the calculations.

The Mauritius publication gives in special detail the mean value for each month and year of the absolute daily ranges. These ranges seem based entirely on hourly readings, and so are not absolutely equivalent to the Katharinenburg ranges dealt with in Table XIII.

As the variation in these daily ranges throughout the year has exceptional features, I give particulars in Table XXIV. There is a resemblance to phenomena at Batavia. In D the variation in the range, though much less conspicuous than in Northern Europe, is well marked; the values for the three midwinter months—May, June, and July are well below the average. In H the variation is small, and somewhat irregular; on the whole, the range is smallest in winter, from May to August, but the next lowest values appear in February and December.

There is a distinct reduction in the V range near midwinter, but a very similar reduction occurs at midsummer.

TABLE XXIV.—Mauritius (Units 1′ for D, 1γ for H and V).

Monthly Means of Absolute Daily Ranges.

	Declination, 1875–80 and 1883–90.	Horizontal force, 1883 to 1890.	Vertical force, 1884 to 1890.
January	6·93	37·9	17·1
February	7·79	35·0	19·5
March	7·11	36·2	20·1
April	5·75	37·6	17·3
May	4·87	35·0	16·5
June	4·03	34·1	15·5
July	4·36	33·8	17·1
August	6·00	34·5	22·0
September	6·28	36·6	22·7
October	6·71	37·4	19·4
November	6·99	37·8	16·7
December	6·78	35·3	15·2
Mean	6·13	35·9	18·2

§ 37. Table XXV. gives the values of a, b, and b/a applicable to the monthly means of the absolute daily ranges at Mauritius. The method of groups was employed, the groups being as follows :—

Years of sun-spot.	For D.	For H.	For V.
Maximum	1880, 1883, 1884, 1885, 1886	1883 to 1886	1884 to 1886
Minimum.	1878, 1879, 1888, 1889, 1890	1887 „ 1890	1887 „ 1890

The seasonal and yearly values of a and b are arithmetic means from the included months, and these means are employed in calculating b/a.

The values of b in Table XXV. fluctuate somewhat erratically from month to month even for D, where 14 years' data are utilised. The exceptional features presented by the September and October data in V had better be regarded provisionally as accidental.

In D and V we have b largest in winter and least at the equinox, a very remarkable feature.

Comparing Tables XXV. and XIII., we see that whilst the mean values of a for the year are fairly similar, the values of b and of b/a at Mauritius are roughly only from a half to a third of the corresponding values at Katharinenburg.

TABLE XXV.—Mauritius (Units 1′ for D, 1γ for H and V).

Monthly Means of Absolute Daily Ranges.

	Declination.			Horizontal force.			Vertical force.		
	a.	10^4b.	$10^4b/a$.	a.	10^3b.	$10^4b/a$.	a.	10^3b.	$10^4b/a$.
January . . .	6·02	381	63	35·1	89	26	14·0	110	78
February . . .	7·34	186	25	29·7	164	55	17·8	55	31
March	6·70	158	23	29·7	200	67	16·8	104	62
April	5·06	278	55	33·5	119	35	14·1	115	82
May	4·10	354	86	30·1	165	55	13·7	97	71
June	3·70	130	35	26·9	209	78	13·2	82	62
July	3·65	290	80	28·3	159	56	13·9	113	81
August . . .	5·20	362	69	28·6	214	75	18·1	157	87
September . .	5·93	152	26	30·8	214	69	20·6	9	4
October . . .	6·41	145	23	31·8	221	70	20·2	[− 47]	[−23]
November . .	6·35	331	52	29·2	376	129	16·2	33	21
December. . .	6·56	120	18	29·7	226	76	13·5	96	71
Winter . . .	4·16	284	68	28·5	187	66	14·7	112	76
Equinox . . .	6·03	183	30	31·5	188	60	17·9	45	25
Summer . . .	6·57	254	39	30·9	214	71	15·4	73	48
Year	5·58	241	43	30·3	196	65	16·0	77	48

§ 38. Table XXVI. deals with the mean of the absolute daily ranges for the year, and with the ranges and the sums of the 24 differences in the mean diurnal inequalities for the year. The grouping of the years is the same as for the previous table. The values of a, b, and b/a for the yearly mean of the absolute daily ranges do not agree quite so closely with the corresponding means of the 12 monthly values in Table XXV. as was the case at Katharinenburg (*cf.* Tables XIII. and XIV.).

In H the values of a and b for the absolute daily range are about double those for the range of the diurnal inequality; in D and V the differences between the two sets of values are smaller, but still considerable. In all three elements the values of b/a for the two species of ranges are fairly similar.

In the case of the mean diurnal inequality in D, the values of b are lower even than those given in Table XIX. for Batavia, and the values of b/a are the lowest we have yet met with. The values of b for the ranges of the mean diurnal inequalities in H and V are much lower than at Batavia, but the values of b/a at the two places are fairly similar.

The values of b/a for the 24 differences do not show that decided excess over the values for the ranges that was seen at Kew and Batavia.

TABLE XXVI.—Mauritius (Units 1′ for D, 1γ for H and V).

	Ranges.						24 differences.		
	Mean of absolute daily values for the year.			From mean diurnal inequality for the year.			From mean diurnal inequality for the year.		
	a.	$b \times 10^4$.	$(b/a) \times 10^4$.	a.	$b \times 10^4$.	$(b/a) \times 10^4$.	a.	$b \times 10^3$.	$(b/a) \times 10^4$.
Declination . .	5·53	255	46	4·06	164	40	15·96	79	49
Horizontal force .	30·4	1859	61	15·0	956	64	116·0	695	60
Vertical force . .	16·2	840	52	11·9	685	58	66·6	292	44

§ 39. Table XXVII. gives the differences between observed and calculated mean yearly data at Mauritius. Comparing the figures in the last line of the table with the corresponding figures in Tables IV., VIII., XV., and XXIII., we conclude that the agreement is not quite so good at Mauritius as at the other stations. The agreement is closest for the mean of the daily declination ranges, where it is very fair; it is on the whole better for V than for H, which is exceptional.

TABLE XXVII.—Mauritius (Units 1′ for D, 1γ for H and V).

Observed Less Calculated Values.

Year.	Mean of absolute daily ranges.			Mean diurnal inequality.					
				Ranges.			24 differences.		
	D.	H.	V.	D.	H.	V.	D.	H.	V.
1875	+0·1	—	—	+0·4	—	—	+ 0·7	—	—
1876	+ ·1	—	—	·0	—	—	+ 0·2	—	—
1877	− ·2	—	—	− ·2	—	—	− 1·8	—	—
1878	+ ·3	—	—	+ ·6	—	—	+ 1·5	—	—
1879	− ·1	—	—	− ·2	—	—	− 0·3	—	—
1880	− ·3	—	—	− ·1	—	—	+ 0·1	—	—
1883	+ ·3	+ 0·3	—	+ ·5	− 0·9	—	+ 0·4	− 2·8	—
1884	+ ·3	− 4·1	+ 0·9	+ ·3	− 3·6	+ 1·1	+ 1·2	− 22·9	+ 6·9
1885	− ·2	− 0·4	− 2·2	+ ·1	+ 1·4	− 1·5	+ 0·7	+ 4·9	− 5·4
1886	− ·2	+ 4 2	+ 1·3	− ·6	+ 3·1	+ 0·5	− 1·4	+ 20·7	− 1·2
1887	− ·2	+ 1·8	− 1·7	− ·6	+ 0·8	− 2·3	− 1·2	+ 9·0	− 6·7
1888	− ·1	+ 2·1	− 0·1	− ·3	+ 1·4	− 0·4	− 0·6	+ 8·5	+ 0·6
1889	− ·1	− 1·4	+ 0·7	+ ·2	− 0·9	+ 1·4	+ 0·8	− 8·0	+ 4·3
1890	+ ·1	− 2·4	+ 1·0	− ·2	− 1·3	+ 1·2	− 0·2	− 9·6	+ 1·9
Mean difference calculated ∽ observed	0·19	2·09	1·13	0·31	1·67	1·20	0·79	10·8	3·9
Probable error	0·14	1·80	·94	0·25	1·41	0·98	0·67	9·2	3·3
Mean value of element	6·11	35·9	18·2	4·44	17·8	13·6	17·8	137	74
Range of element	1·90	13·2	6·8	1·90	7·0	6·8	7·0	46	28
Mean difference × 100 / mean value	3	6	6	7	9	9	4	8	5
Probable error × 100 / range of element	7	14	14	13	20	14	10	20	12

Summary.

§ 40. A slight progressive decrease in a and increase in b in the sun-spot formula (1) is suggested by the Greenwich D and H data from 1841 to 1896, but this does not meet with support from Signor RAJNA's analysis of D ranges at Milan from 1836 to 1894. In both cases there is an element of uncertainty, arising from want of homogeneousness in the data.

According to RAJNA's earlier data, values of a, and to a lesser extent values of b, calculated from periods as long as 14 years may differ very sensibly from those calculated from longer periods, but differences of this kind seem to have diminished since observations became more homogeneous and are probably ascribable in part to observational uncertainties.

Results calculated for Milan from the period 1890 to 1900, which is the period chiefly utilised in the present paper, differ but little from those found by RAJNA for the periods 1836 to 1894 and 1871 to 1894.

The tendency for b to be small in winter and large in summer, described at Kew, is also, in general, conspicuous elsewhere; but there are exceptions, especially at tropical stations.

The tendency in b/a to be large in winter as compared to summer, so prominent at Kew, is also, in general, prominent at other northern stations, but the phenomenon is comparatively inconspicuous in the case of the declination range at Greenwich. At the tropical stations the seasonal change in b/a appears much reduced and is somewhat uncertain.

There is no conspicuous difference between the "all" and the "quiet" days' mean yearly values of b and b/a for the ranges of the D and H diurnal inequalities at either Greenwich or Pawlowsk; but at Pawlowsk there is a somewhat notable difference between "all" day and "quiet" day D results in winter, and the difference between "all" and "quiet" day V results is very large throughout the whole year.

If we exclude Mauritius, the values of $10^4 b/a$ for the ranges in the mean diurnal inequality of declination for the year at the several stations vary only from 65 to 73. The corresponding values of b show also a pretty close agreement at the northern stations, but the values for the tropical stations are much smaller.

In H there is no very conspicuous difference in the values of b or b/a for the ranges from the mean diurnal inequality for the year at the northern stations; but the values found for b/a at Batavia and Mauritius are considerably smaller, while the value found for b is smaller at Mauritius, but very materially larger at Batavia.

When the formula (1) is applied to any ordinary measure of magnetic disturbance, it gives much too high values for 1893—the year of sun-spot maximum—and much too low values for 1892. Thus the application of (1) to disturbances has not the same justification as its application to ordinary diurnal inequalities. It may, however, serve a useful purpose in giving a greater degree of definiteness to the comparison of contemporaneous disturbance phenomena at the same or at different stations.

In the case of results obtained by the application of (1) to individual months of the year a considerable latitude must be allowed to chance, especially in winter months when the diurnal range is small, unless an exceptionally long series of observations is available. Results obtained from arranging months in seasons are much less exposed to numerical uncertainties, but they are insufficient for the reason that there are conspicuous differences between months which have to be grouped under the same season. This remark applies more particularly to winter and equinoctial months in higher latitudes.

[*June* 8, 1904.—The following additional data—all obtained by the method of least squares—apply to the ranges of the mean diurnal inequalities for the year at Irkutsk ("all" days) and Colaba ("quiet" days), and to the mean difference between the absolute daily maximum and minimum at Zi-ka-wei. The units are 1′ for angles, 1γ for force components.

Place . . .	Irkutsk (Siberia).			Zi-ka-wei (China).			Colaba (Bombay).		
Latitude . .	52° 16′ N.			31° 12′ N.			18° 54′ N.		
Longitude . .	104° 16′ E.			121° 26′ E.			72° 49′ E.		
Period of years	1890 to 1900.			1890 to 1900.			1894 to 1901.		
Element . .	a.	10^4b.	$10^4b/a$.	a.	10^4b.	$10^4b/a$.	a.	10^4b.	$10^4b/a$.
D.	4·815	358	74	4·369	303	69	2·373	66	28
I.	0·971	87	90	—	—	—	—	—	—
H.	18·18	1896	104	—	—	—	31·65	2814	89
V.	6·49	710	109	—	—	—	19·35	723	37

At Irkutsk the values of b for D and H are similar to those at Katharinenburg; the values of b/a for these elements are similar to those at Kew; in V the values of b and b/a are decidedly less than at Katharinenburg.

At Zi-ka-wei b/a is decidedly less, and b much less, than the corresponding values for Katharinenburg (second line of Table XIV.).

At Colaba the ("quiet" day) values of b and b/a for D are notably less than the corresponding ("all" day) values at Mauritius, the smallest occurring in the paper; but the value of b for H exceeds that at Batavia, the largest previously noted. The value of b/a for V is conspicuously small.]

Solutions.

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Saturated Solutions, Physical Constants of.
BERKELEY, Earl of. Phil. Trans., A, vol. 203, 1904, pp. 189–215.

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VIII. *On some Physical Constants of Saturated Solutions.*

By the Earl of Berkeley.

Communicated by F. H. Neville, *F.R.S.*

Received March 28,—Read May 19, 1904.

Introduction.

The following work was undertaken with a view to obtaining data for the tentative application of Van der Waals' equation to concentrated solutions. It is evidently probable that if the ordinary gas equation be applicable to dilute solutions, then that of Van der Waals', or one of an analogous form, should apply to concentrated solutions—that is, to solutions having large osmotic pressures.

Saturated solutions were taken for investigation because they presumably have the greatest osmotic pressures, and also because there is reason to believe that, in concentrated solutions at a given temperature, the greater the concentration the less the relative dissociation. For the purpose in view, measurements of volume, pressure and temperature are required.

Volume.

This term of the equation is deducible from observations of the density of a saturated solution and of the solubility of the salt at varying temperatures. In Part I., I give the densities and solubilities obtained, together with a description of the methods and apparatus used.

Pressure.

I am at present engaged, with the collaboration of Mr. E. G. Hartley, in testing a method of directly observing large osmotic pressures. Should the method fail, I propose to determine the vapour pressures of the saturated solutions at different temperatures and from these calculate the corresponding osmotic pressures. The observations and details will be given in Part II.

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Temperatures.

The temperatures at which the densities, solubilities and osmotic pressures were determined are given with those quantities respectively.

Part III. will be devoted to the application of the results to theory.

The selection of the particular salts whose solutions were examined was governed by the following considerations :—

(1.) Fairly soluble salts should be used, so that differences between the ordinary phenomena of dilute and those appertaining to concentrated solutions may be the more marked ;

(2.) They should have as wide a range of molecular weights as possible, so as to bring into prominence any effect the interacting masses may have on the space occupied by the molecules ;

(3.) For the purpose of comparing members of the same family of elements the salts should be isomorphous, the presumption being that isomorphous salts give similarly constructed molecules in solution.

PART I. (A).

Determination of the Constants.

The densities were obtained by the following method : An approximately saturated solution was kept in contact with crystals of the salt at a definite temperature by means of a thermostat, and continuously stirred. When a sufficient length of time had elapsed, a pyknometer, whose capacity was known, was immersed in the solution and filled to a mark, then washed and dried by means of pure alcohol, and weighed against a counterpoise which had been similarly washed and dried. The solubilities were determined by washing the contents of the pyknometers into platinum crucibles and weighing them after evaporating to dryness. It was found in the course of the work that, in the case of very soluble salts, this method was not satisfactory, because a crust of salt formed on top of the solution in the platinum crucible, and the accumulation of steam under it, on finding its way out, carried particles of solution with it. Glass bulbs, represented in fig. 1 and made of Jena glass, were therefore substituted for the platinum crucibles and the solution evaporated to dryness in them. This was effected by passing a current of dry air through the tubes while they were being heated to 110°–170° in an air oven, the air current and the

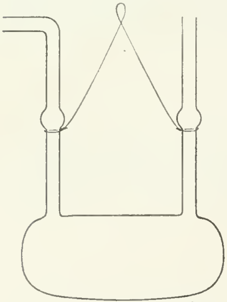

Fig. 1.

heating being continued until the bulbs had attained a constant weight. The air current was obtained by means of a Fleuss pump. The same filling of a pyknometer gave, therefore, both the density and the solubility. As a check on the latter, the contents of a pyknometer were occasionally analysed.

Pyknometers.

At first Sprengel pyknometers of various shapes and sizes were tried, but were found to be unsatisfactory. This was because it was almost impossible to prevent the solution from crystallizing in the capillary during the time the level of the liquid was being adjusted to the mark.

The following was the form finally adopted and found quite satisfactory. A pear-shaped bulb, of about 5 cub. centims. capacity, terminating above in a stem composed of a graduated capillary 120 millims. long, and below in a finer capillary, bent as in fig. 2, was used for salts of medium solubility. For somewhat insoluble salts a similar pyknometer, but of about 11 cub. centims. capacity, was found to be more suitable, while for very soluble salts, such as sodium sulphate, which have great differences in solubility at different temperatures, it was necessary to have similar pyknometers made of thicker glass, so that when the crystals formed and practically filled the whole of the bulb, the latter would withstand the pressure. It was also found necessary to make the capillaries of a larger internal diameter, so as to be able to fill quickly. And the shape of the lower capillary (see fig. 3) was altered and its end fitted with a glass cap to prevent the solution from "creeping" out when on the balance. The stem was also fitted with a cap to prevent evaporation.

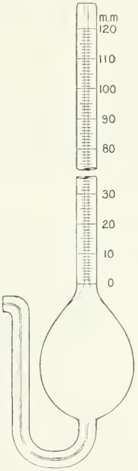

Fig. 2.

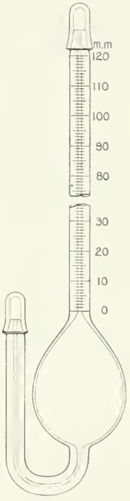

Fig. 3.

Determination of the Capacities of the Pyknometers.

Before determining the capacities, the pyknometers were heated rapidly and repeatedly to 200° C., being allowed to cool to the temperature of the room between each heating; by this means it was hoped that the gradual shrinking in volume would be accelerated. The graduated capillaries were then calibrated by the usual method of running a thread of mercury along the bore, measuring its length, and then weighing it.

The capacities were found by weighing the pyknometers filled with water at 0° C. and at 90° C. respectively; the volume occupied by the water was taken to be that given in LANDOLT and BÖRNSTEIN's tables for water which is air-free. The difference between the capacities thus determined gave the expansion from 0° C. to 90° C., and for intermediate temperatures it was assumed to be proportional to the temperature interval; this assumption was tested with one of the pyknometers, and it was found that the resulting difference was within the experimental errors. With the 11 cub. centim. pyknometers, however, it was deemed advisable to examine the error more closely, and for this purpose the capacities were determined at five approximately equal intervals of temperature between 0° C. and 90° C. The numbers obtained were plotted against the corresponding temperatures, and a bent-ruler curve passed through the points; the capacities for intermediate temperatures were taken from it. The maximum difference between this curve and a line joining the penultimate observations represented a difference of ·0015 cub. centim. This is a quantity which is barely greater than the experimental errors, as will be seen from the following numbers obtained with one of the pyknometers:—

Temperature.	Capacity.	Temperature.	Capacity.	Temperature.	Capacity.	Temperature.	Capacity.	Temperature.	Capacity.
° C.	cub. centims.	° C.	cub. centims.	° C.	cub. centims.	° C.	cub. centims.	° C.	cub. centims.
91·85	11·4406	68·60	11·4321	45·40	11·4239	25·85	11·4181	0·70	11·4111
91·35	·4397	68·40	·4317	45·30	·4239	25·80	·4186	0·60	·4113
91·10	·4394	68·00	·4313	45·15	·4239	25·40	·4176	0·50	·4111
—	—	—	—	45·30*	·4238*	—	—	—	

On re-determining the capacities after an interval of several months no change was apparent.

As the table used for the expansion of water gives numbers derived from air-free water, and as the pyknometers had been filled with water which had not been freed from dissolved air, it was thought possible that an error had been introduced in this way; a pyknometer was therefore filled, in a vacuum, with water which had been boiled in that vacuum for three-quarters of an hour; it was then withdrawn and brought to a constant temperature in the thermostat and weighed in the usual manner. The results of three observations carried out thus did not differ from those obtained with ordinary water by more than the latter differed among themselves. Taking into consideration that the solutions themselves are not air-free, it was considered unnecessary to pursue the matter any further.

* This observation was one made with air-free water.

In weighing the pyknometers care was always taken that they should be slung on the balance in such a way that the end of the lower capillary was at a higher level than the level of the liquid in the stem; this, as a rule, was sufficient to prevent any loss of weight by evaporation, but such observations as did show a loss were rejected.

Stirring.

The stirring was obtained by means of a small platinum rod, fitted with a two-bladed screw, suspended vertically in the solution, and rotated by a cord and a small electric motor. The speed of rotation could be varied from 2 to 20 revolutions per second. During the last two years of the course of this work the separate motors were replaced by a shafting driven by an electric motor; pulleys of various sizes were fixed on it and driving cords taken to the stirrers as required. This shafting also worked the Fleuss pump mentioned above.

Constant Temperatures.

At 0° C. the beaker containing the solution was surrounded by ice and water. At 15° C. it was placed in a copper vessel, in which was suspended a thermostat, and through which a current of cold water passed. The former actuated a gas burner and kept the temperature constant. From 30° C. to 90° C. the solution was placed in a platinum beaker of 300 cub. centims. capacity which formed part of a D'Arsonval thermostat. The platinum beaker (see fig. 4), having a flange welded on to it three-

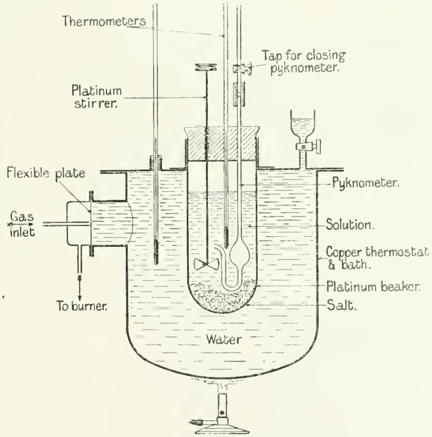
Thermometers
Tap for closing pyknometer.
Platinum stirrer.
Flexible plate
Gas inlet
To burner.
Pyknometer.
Solution.
Copper thermostat & bath.
Platinum beaker.
Salt.
Water

Fig. 4.

quarters of an inch from the top, was passed through the top plate of the thermostat and soldered in position. The body of the thermostat was of copper and held about three litres of water, and the expansion and contraction of this was enough to actuate the flexible diaphragm (the iron plate of the receiver of a telephone) sufficiently to keep the temperature of the thermostat constant to about 0°·2 C.; that is to say, that for 2 or 3 hours before taking an observation, the temperature in the beaker would only show, at the utmost, a change of 0°·05 C., but in the course of 24 hours after setting the temperature of the thermostat, and consequently that of the solution, might slowly rise or slowly fall to the extent mentioned, and then remain constant to 0°·05 C.

Means employed for Determining the Point of Saturation.

After numerous experiments the following general method was found to be the most satisfactory. The thermostat was set at 90° C., and water, together with a quantity of salt more than necessary to saturate it, placed in the platinum beaker, and the mixture stirred very rapidly until it was thought that saturation had been attained; an observation of the density was then made, and the stirring continued for a further period of 2 or 3 hours, and then another density determination made. If the two observations agreed within the experimental errors, it was considered that saturation had been practically reached; if the two observations did not agree, the stirring was continued and the density taken at intervals until it became constant. The temperature of the thermostat was then lowered 2 degrees, and after stirring 2 or 3 hours the density again taken. The mean of this and of the constant density previously mentioned was considered to be the density of a saturated solution at the mean of the respective temperatures. The temperature of the thermostat was then lowered to the next point of observation, and after 2 or 3 hours' stirring the density was taken; water was then added to the solution and the stirring continued until the density, taken at intervals of from 4 to 12 hours, was constant—the mean of the first and the last observations, which usually differed by an amount slightly greater than the experimental errors, was taken as giving the true density. The process was then repeated for the other temperatures. It should be noted that whether working with supersaturated or an unsaturated solution, the liquid is always stirred in contact with a large excess of solid salt.

In the case of salts whose solubilities decrease with an increase of temperature, the process is reversed; with Na_2SO_4, for example, which has a maximum solubility at 32°·5 C., the thermostat was set at 33° C., and stirring continued until constant density was obtained; the temperature was then raised 1° C. and the density again determined, and the mean of this and of the constant density above mentioned was taken as the true density of a solution saturated at the mean of the respective temperatures. The temperature of the thermostat was then raised to the next point of observation, the density taken, boiling water added and a constant density obtained, the numbers being "meaned" as before. The process was then continued for the next higher temperature, and so on. The object of adding boiling water is to make sure that the solution is unsaturated, for if cold water were added the temperature of the solution would fall, and if the rate of attaining saturation be greater than the rate at which the solution comes back to the constant temperature, you get a solution supersaturated with respect to that temperature.

It will therefore be seen that the method adopted resolves itself into this: at any given temperature, two observations of density and solubility are taken; one is obtained by stirring a supersaturated solution in contact with the solid salt, the other by stirring an unsaturated solution in contact with an excess of salt—and the true density or solubility is considered to be the mean of the two observations.

In the earlier part of this work it was found that, in many cases, a very long time elapsed before the densities obtained, when starting with an unsaturated solution, approached sufficiently closely to that derived by starting with a supersaturated one—this was partially remedied by increasing the speed of stirring from 2 to 20 revolutions per second—but even then there was generally a difference in the two densities of some few units in the 4th decimal place. The cause of this discrepancy was eventually traced to the fact that a considerable length of time was also required for the point of saturation to be attained by a supersaturated solution, even when stirred in contact with its salt. It was owing to this that some 300 density and solubility determinations had to be discarded—for preliminary observations had shown that concordant results could be obtained by merely covering the top of the beaker with a glass perforated for the stirrer to pass through, and removing the plate while the pyknometer was being filled. At the higher temperatures the removal of the plate caused a fall in the temperature of the solution and a consequent supersaturation. This, however, was not suspected (because when the results were plotted the curve was regular) until I was dealing with very soluble salts, which, on the removal of the plate, tended to form crusts of salt on the surface of the solution. The difficulty was overcome by closing the beaker by an india-rubber stopper, which was perforated for the stirrer, the thermometer, and the pyknometer. The latter was closed at the upper end by a tap attached by rubber tubing. The tap served two purposes: it was kept closed on immersing the pyknometer, so that no liquid could enter during the time that the pyknometer was attaining the temperature of the solution, and it was closed after filling the pyknometer, so that no liquid could flow back during the withdrawal of the rubber stopper.

As an extreme example of the necessity of giving an unsaturated solution plenty of time to attain saturation, and also as showing the importance of having a sufficiency of salt in contact with the solution, I extract the following numbers from my note-book. An unsaturated solution of thallium alum, together with a quantity of the salt, was placed in the beaker, which was at the constant temperature of 61°·0 C. This was stirred for 12 hours at the rate of 10 to 20 revolutions per second; at the end of this period the density was found to be 1·2539 (temperature 61°·00 C.). Having reason to believe, from previous work, that saturation had been reached, the temperature of the thermostat was lowered 1° C. and the solution stirred for another 3 hours; its density was then 1·2546 (temperature 59°·85 C.).

The next day it was 1·2555 (temperature 59°·90 C.).
,, ,, ,, 1·2572 (,, 60°·00 C.).
,, ,, ,, 1·2591 (,, 60°·00 C.).

Between each observation about 10 hours' continuous stirring was given to the solution, and all the time there had been about 5 cub. centims. of solid salt in contact with the solution; another 20 cub. centims. of salt was then added, and the stirring continued for 12 hours, with a resulting density 1·2810 (temperature 60°·00 C.). And a further 12 hours gave 1·2813 (temperature 60·00 C.).

On the other hand, the following shows the reverse phenomenon, *i.e.*, that a considerable time must elapse before a supersaturated solution attains its true point of saturation. A solution of Na_2SO_4, saturated at 60° C., was heated to the constant temperature of 75° C. (it must be remembered that Na_2SO_4 is more soluble at 60° C. than at 75° C.) and stirred at the rate of 13 revolutions per second in contact with the anhydrous salt for $3\frac{1}{2}$ hours; the density of the solution was found to be 1·2738 (temperature 75°·00 C.). The next day, after 12 hours' stirring, the density was 1·2729 (temperature 75°·00 C.); 20 cub. centims. of boiling water was then added (if cold water had been added, as before explained, the solution would have become supersaturated), and after 12 hours' stirring its density was 1·2727 (temperature 75°·00 C.).

Where it was suspected that the solutions, when at the higher temperatures, might decompose non-reversibly, the observations for the lower temperatures were first recorded, in the manner already outlined, and those for the higher temperatures were obtained by heating to the constant temperature required and stirring the solution until the density was constant; the temperature was then lowered by 1 or 2 degrees, and after a sufficient length of time the density again determined; the mean of this last observation, and of the constant density first obtained, was considered to be the density of a solution saturated at the mean of the corresponding temperatures.

The following are the important points to be observed in obtaining a saturated solution:—

(1) A sufficiency of solid salt should always be in contact with the solution;
(2) A thorough stirring should be continuously kept up;
(3) A sufficient length of time should be allowed to elapse before taking the required observation. This last condition seems to depend on the nature of the salt, the speed of stirring, and on the temperature.

I had hoped to have been able to determine both the rate of attainment ot saturation, and the time at which it is attained, by observing the change in the electric conductivity of the solution while it is becoming saturated, and I have made a few somewhat unsatisfactory experiments on the method, but hope to be able to return to it shortly.

Measurement of Temperature.

Thermometers whose graduations were sufficiently open to allow of an estimation to 0°·01 C. were used. They were standardized at Kew, and the rise of the zero point was determined after an interval of 18 months. The rise of the zero point was assumed to be proportional to the elapsed time, and all observations are corrected on that assumption. Corrections were also applied for the emergent column by suspending an auxiliary thermometer half way up the exposed stem and calculating in the usual manner; in no case did this correction amount to more than 0°·37 C.

The temperatures given in the tables are corrected to the hydrogen scale of the Bureau International at Paris.

Experimental Errors.

On page 192 I have already given an example of the results obtained in determining the capacities, and it will be seen that the largest difference between any two observations at the same temperature is ·0012 cub. centim., which is roughly ·01 per cent. To give an idea of the order of accuracy of the observations of density and solubility, the actual figures obtained with NaCl are appended below. NaCl was selected because the differences between the two sets of densities are fairly typical, while, on account of the small change in the solubility of the salt, those between the solubilities, besides being typical, show the experimental errors without the necessity of correcting for small changes of temperature.

Starting supersaturated.			Starting unsaturated.		
Temperature.	Density.	Solubility.	Temperature.	Density.	Solubility.
° C.			° C.		
0·35	1·20900	35·75	0·35	1·20896	35·74
15·05	1·20209	35·83	15·35	1·20193	35·85
30·05	1·19556	36·22	30·05	1·19555	36·19
45·30	1·18908	36·62	45·50	1·18902	36·59
61·80	1·18221	37·30	61·60	1·18227	37·26
75·85	1·17644	37·86	75·45	1·17637	37·80
90·50	1·17009	38·53	91·25	1·16971	lost

The largest difference between two densities at the same temperature is ·00038 at 91° C., which, if the observation at 90°·5 C. be corrected to 91°·25 C., is reduced to a difference of ·00020, and this corresponds to an error of about 0·02 per cent. In solubilities, however, the largest difference is ·06 at 75° C., and this corresponds to an error of 0·16 per cent. This large difference in the two percentage errors is remarkable, and I have not yet been able to account for it—it is manifested in most of the salts hitherto worked with. A fact which may possibly throw light on the subject is noticeable in the above table, and is one which most of the salts also show, namely, that the solubilities obtained when starting with an unsaturated solution, tend to be slightly less than those obtained when starting with a supersaturated one, and this although the corresponding densities are practically identical. I hope to investigate the matter while determining the electric conductivities of these solutions.

The Densities and Solubilities at the Boiling-point.

Attempts were made to determine these in a Beckmann apparatus, but without success—the difficulty of keeping a constant temperature being too great—so recourse was had to a method first suggested by, I believe, Buchanan.

In the apparatus shown in fig. 5, the outer glass tube A contains water, and the inner tube B the salt and solution; by boiling the water vigorously and closing the side tube C, steam, passing through the tube D, is forced to bubble rapidly through the solution (D is graduated in centimetres so that the level of the solution may be estimated while the pyknometer is in the solution). The steam, if passed rapidly enough through the solution, stirs it thoroughly, and the temperature rises up to the boiling-point of the saturated solution and remains constant at this point as long as there is enough undissolved salt left. The constancy of the temperature therefore indicates that saturation is attained.

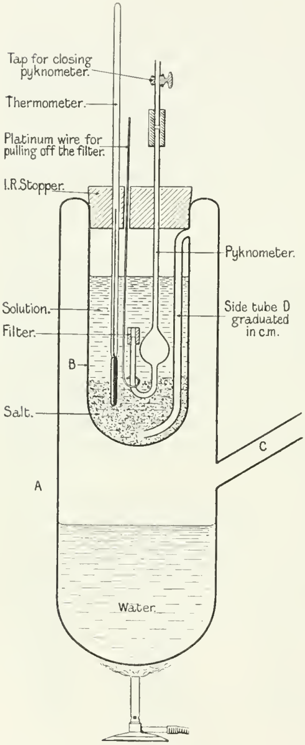
Tap for closing pyknometer.
Thermometer.
Platinum wire for pulling off the filter.
I.R.Stopper.
Pyknometer.
Solution.
Filter.
Side tube D graduated in c.m.
B
Salt.
C
A
Water.

Fig. 5.

Determination of the Density.

When it is seen that the steam is passing freely through the solution, an india-rubber plug, through which the thermometer and pyknometer pass, and which is also perforated to allow the steam to escape, is inserted into the top of the inner tube. When the temperature becomes constant, the pyknometer, with the tap closed and with the end of the lower capillary covered by a filter, is forced through the stopper, so that the bulb and capillary are completely immersed; when the temperature is again constant, the tap is opened and the pyknometer quickly filled by gentle suction, and the tap closed. The filter is then removed from the end of the capillary, the level of the solution in the stem is read, and both thermometer and pyknometer are taken out of the solution by the withdrawal of the rubber stopper; the pyknometer is then washed, dried, and weighed in the usual manner.

It was found advisable to use the pyknometers described on p. 191 and shown in fig. 3, not only on account of the pressure set up when the salt crystallized out, but because they could be more quickly filled, and therefore less condensed steam formed in the stem; a further reason for using these pyknometers was that the larger bore of the stem and lower capillary enabled them to be emptied with less difficulty.

Determination of the Solubility.

When the pyknometer had been weighed, its contents were emptied into a beaker, and the solution washed into a Jena glass bulb (described on p. 190 and shown in fig. 1), and evaporated to dryness as before.

Great difficulty was experienced in emptying the pyknometers when filled with the solutions of rubidium nitrate, thallium nitrate, or cæsium alum, and the only way of doing so was by alternate heating and cooling when completely immersed in nearly boiling water—the operation taking in some cases as long as 6 hours. It is interesting to note that on testing the boiled saturated solutions of the nitrates of sodium and rubidium for nitrites by means of fuchsine, the conversion of a small quantity of the nitrate into the nitrite was distinctly indicated.

Modification of Apparatus Necessary to Meet the Case of Extremely Soluble Salts.

In the case of the nitrates of rubidium and thallium, which are extremely soluble at the boiling-point, the apparatus described on the foregoing page was found to be unsuitable, because a constant temperature could not be maintained for a sufficient length of time to allow the pyknometer to be filled. Two things are essential for maintaining the solutions at their boiling-points: that thorough stirring should take place, and that there should be a sufficiency of undissolved salt left in contact with the solution; with extremely soluble salts the larger quantity of steam necessary for thorough stirring dissolves so much salt that by the time this stirring is attained the solution is nearly clear, and shortly after, all the salt is dissolved and the temperature begins to fall.

A modification of the method was adopted in which steam, generated in a boiler A (see fig. 6), is forced through a tube B and delivered at the bottom of the large test-tube C, which contains the solution. The test-tube is immersed in an oil bath D maintained at a temperature close to that of the boiling-point of the saturated solution, the oil in the bath being vigorously stirred by a stirrer driven from the main laboratory shafting. When the temperature of the oil bath was below the boiling-point, salt dissolved; when above, salt was thrown out of solution. By care-

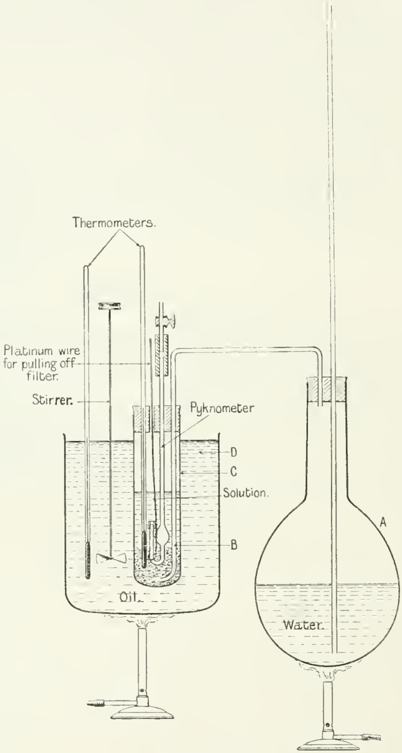
Thermometers.
Platinum wire for pulling off filter.
Stirrer.
Pyknometer
D
C
Solution.
B
A
Oil.
Water.

Fig. 6.

fully adjusting the temperature of the oil bath, two densities could be obtained, one while the temperature of the solution was close to the boiling-point, but slowly rising, and the other when it was above the boiling-point, but close to it and slowly falling. The former gave the density of a slightly unsaturated solution, and the latter that of a slightly supersaturated solution when referred to a solution saturated at its boiling-point. The mean of these two observations was considered to be the density of the solution saturated at the boiling-point—and similarly with the resulting solubilities.

The results obtained at the boiling-point were found not to be as concordant as those at the other temperatures; doubtless the greater part of this is due to the exceptional difficulties of the experiments. The following are some of the sources of error. It was impossible to prevent the condensation of steam in the stem of the pyknometer, and it was therefore necessary to estimate the length of each drop, and add this length to the reading of the level in the stem. There was also an error introduced by the fact that, for the purpose of reading its level, the solution had to be sucked into the cold part of the stem which projected through the indiarubber stopper; on reaching this colder part, salt immediately crystallises out; the total volume thus changes, and the observed level is not that which the solution would otherwise have attained. The maximum error possible from this cause was calculated for the case of rubidium nitrate, and was found to be such as to give an error in the density of 0·1 per cent.

Another and a much more important source of error was that the reduction of pressure on the surface of the solution in the pyknometer, unavoidable when filling by suction, very often caused steam bubbles to form. As it was essential to fill whilst a large excess of undissolved salt was still being stirred by the steam, the solution surrounding the pyknometer was often semi-opaque, and consequently it might happen that part of the space inside the pyknometer was occupied by an unseen steam bubble, and might thus be an unobserved source of error. Numerous fillings had to be rejected on this account, the steam bubbles showing themselves on the withdrawal of the pyknometer from the solution. The numbers tabulated below are those derived from fillings in which there were no observed steam bubbles.

Determination of the Temperature.

As mentioned above, the boiling-point of the saturated solution was considered to be the constant temperature which the solution and salt reached when steam was rapidly bubbled through them; this temperature was indicated by mercury thermometers, and is given in the table of results. They are, however, uncorrected for emergent column, because it was found to be practically impossible to apply a satisfactory correction. It is hoped that, later on, when determining the osmotic pressures,* these boiling-points will be accurately ascertained by means of platinum thermometers. In the expectation of this the total pressures under which each solution was boiling when its density was taken was noted, and is given in the table. This total pressure is made up of the barometric pressure, together with the pressure due to the height of the boiling liquid. To ascertain the effect of the latter, observations were made on the boiling-point of water of varying depths, and through which steam was being rapidly blown. The results showed that the boiling-point was increased by an amount equal to that which a pressure equal to half the depth of the liquid would create; this, of course, was what was to be anticipated, provided the stirring was thorough. It was assumed that a similar result would hold for the solutions, and the total pressures given are those calculated on this basis.

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* These experiments are in progress, but not complete.

Results.

The first table gives the results obtained by means of the apparatus shown in fig. 5.

Column I. gives the approximate boiling-point, which is also the temperature at which the pyknometer was filled.

Column II. gives the total pressure in millimetres of mercury, at the time of filling.

Columns III. and IV. give the corresponding densities and solubilities; the latter are in parts of anhydrous salt dissolved by 100 parts of water.

TABLE I.

	I. Boiling-point.	II. Pressure.	III. Density.	IV. Solubility.
	° C.			
NaCl	107·5	740·1	1·1634	39·57
,,	107·8	749·1	1·1629	39·72
KCl	107·4	738·7	1·2118	58·09
,,	107·4	739·4	1·2118	58·12
RbCl	112·9	756·6	1·6146	146·65
,,	112·9	756·6	1·6149	146·65
CsCl	119·3	754·2	2·0855	290·04
,,	119·5	757·6	2·0863	289·93
TlCl	99·2	736·5	·9787	2·42
,,	99·6	746·9	·9786	2·40
Na_2SO_4	101·9	750·6	1·2451	42·15
,,	101·9	751·0	1·2449	42·21
K_2SO_4	101·0	752·6	1·1206	24·23
,,	101·0	752·6	1·1207	24·18
Rb_2SO_4	102·4	742·4	1·4752	82·57
,,	102·4	742·4	1·4753	82·56
Cs_2SO_4	108·5	737·2	2·0927	224·24
,, *	108·6	737·6	2·0942	224·75
Tl_2SO_4	99·7	748·7	1·1164	18·45
,,	99·6	747·5	1·1165	18·45
KNO_3	114·0	745·5	1·6266	311·79
,,	114·0	745·1	1·6272	311·48
$CsNO_3$	106·0	747·6	1·8642	219·29
,,	106·3	749·2	1·8664	221·12

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* These observations are derived from pyknometer fillings which were considered to be particularly good; they are therefore given double weight when taking the "means" for the tables at the end of the paper.

The second table gives the results obtained with the apparatus shown in fig. 6.

Columns I., II,, III., and IV. give the temperature of filling, the total pressure, the density, and the solubility, respectively, when the temperature of the oil bath was below the boiling-point, but close to and rising; while columns V., VI., VII., and VIII. give the same, when the oil-bath temperature was higher than the boiling-point, but close to and falling.

TABLE II.

	I. Temperature.	II. Pressure.	III. Density.	IV. Solubility.	V. Temperature.	VI. Pressure.	VII. Density.	VIII. Solubility.
	° C.				C.			
$NaNO_3$	119·0	737·7	1·5369	208·27	118·9	734·8	1·5379	209·42
$RbNO_3$	118·1	739·6	2·1867	614·27	118·4	729·2	2·1867	619·94
$TlNO_3$	104·1	756·8	3·1725	583·39	104·8	768·1	3·2086	604·46
CsAlum	100·3	757·8	1·1278	22·47	100·4	758·0	1·1292	23·21

Purity of the Salts—the Chlorides.

The alkali chlorides were obtained from Messrs. MERCK, and were sold as the purest they made; the thallous chloride came from Messrs. KAHLBAUM, and was also sold as pure.

The solutions of the potassium and sodium salts did not require filtering, and were tested for purity by an analysis of their chlorine contents. The sodium salt giving 60·58 per cent. (calculated 60·59 per cent.) and the potassium 47·60 per cent. (calculated 47·54 per cent.). The rubidium chloride was tested spectroscopically for the presence of potassium and cæsium by first locating the chief lines of these metals by observation of their spectra on the graduated circle of the spectroscope, and then exploring the rubidium spectrum for them. No definite evidence of impurities was obtained. An analysis of the chlorine content gave 29·34 per cent. (calculated 29·32 per cent.). From the appearance of the cæsium chloride it was thought necessary to filter the solution and recrystallise several times; the mother liquor of the first recrystallisation was distinctly yellow, that of the second faintly so, while the third was colourless. A spectroscopic examination, similar to that mentioned above for the rubidium salt, revealed, it was thought, a trace of rubidium. An analysis of the chlorine content gave 21·13 per cent. (calculated 21·06 per cent.).

The thallous chloride was found to be free from lead, and an analysis of the thallium content gave 85·40 per cent. (calculated 85·21 per cent.). Owing to the insoluble nature of this salt, the solubility determination cannot be relied on to as great a degree of accuracy as in the other determinations.

The Sulphates.

The alkali sulphates came from Messrs. Merck, and were sold as their purest; the thallium salt came from Messrs. Kahlbaum. Neither the sodium nor the potassium salts required recrystallising, nor did their solutions require filtering; analyses of their sulphuric acid contents gave for the sodium salt 67·37 per cent. (calculated 67·57 per cent.) and for the potassium 55·20 per cent. (calculated 55·12 per cent.). Not having purchased enough of the rubidium salt, the balance was made good by treating pure rubidium carbonate (also purchased from Messrs. Merck) with pure sulphuric acid in just sufficient quantity to neutralise the solution, and then crystallising out. The two quantities of salt were then added together and recrystallised, and the crystals examined spectroscopically, in the manner before stated, for potassium and cæsium, but with no definite indication of either. An analysis of the sulphuric acid content gave 36·05 per cent. (calculated 35·99 per cent.).

The cæsium sulphate was recrystallised three times, and the spectroscopic examination gave no definite indication of either potassium or rubidium. An analysis of the sulphate content gave 26·62 per cent. (calculated 26·55 per cent.). The thallium sulphate was recrystallised three times and found to be free from lead. An analysis of the thallium content gave 80·96 per cent. (calculated 80·95 per cent.).

The Nitrates.

All the salts were Messrs. Merck's purest, except the thallium salt, which came from Kahlbaum. The alkali nitrates were all recrystallised two or three times, and were examined spectroscopically and found to be free from impurities. The thallium nitrate, however, was found to contain some lead; it was freed from this by repeated recrystallisation. An analysis of the thallium content gave 76·89 per cent. (calculated 76·69 per cent.).

On account of the difficulty of obtaining accurate analyses of the alkali nitrates they were not analysed, but after the first recrystallisation a series of densities and a corresponding series of solubilities at different temperatures were obtained, and these series were compared with similar series obtained from the solution of the crystals of the next recrystallisation. The two differed by no more than the experimental errors.

During the evaporation to dryness in the Jena glass bulbs for the purpose of determining the solubilities, it was found that a trace of nitrate almost invariably came over with the condensed water, and those observations in which more than a trace came over were rejected. It was also noticed that, except in the case of cæsium nitrate, the dried salt remaining in the bulbs contained a trace of nitrites. The quantities in both cases were so small that it was not considered necessary to apply any corrections to the resulting solubility.

The Alums.

The potassium alum purchased as pure from Messrs. MERCK was found to contain a small quantity of both iron and ammonium, and repeated recrystallisation did not purify it. Pure aluminium and pure potassium sulphates were therefore purchased, and the pure alum made from these. An analysis of the sulphuric acid content of this salt, when recrystallised, gave 40·33 per cent. (calculated 40·49 per cent.). The remainder of the alums, purchased as pure from Messrs. MERCK, were recrystallised several times, and the spectroscopic examination showed no impurities. Analyses gave, for the rubidium salt, 36·69 per cent. SO (calculation being 36·89 per cent.), and, for the cæsium salt, 33·67 per cent. SO (calculation being 33·80 per cent.).

The thallium alum was analysed by determining the thallous sulphate content, giving 39·29 per cent., the calculated value being 39·43 per cent. It will be noticed that the tables of results give no values for potassium alum above 60° C., for the rubidium alum above 70° C., and for thallium alum above 60° C.—this is because it was found that the prolonged heating at 68° C., at 80° C., and at 75° C. respectively decomposed the solutions, and a white insoluble precipitate was formed. The cæsium alum, however, could be heated to the boiling-point without decomposition.

The solubilities of the potassium and rubidium alums could not be determined to so close a degree of accuracy as that of the other salts, for it was almost impossible to dry the contents of the pyknometers to a constant weight, without decomposing the salt. The method finally adopted was to evaporate to partial dryness in the Jena glass bulbs at 115° C., then raise the temperature of the oven gradually to 175° C. (dry air passing the while), and, when all perceptible moisture had been driven off, to heat the bulb gently with a naked flame, care being taken not to heat to a temperature high enough for the glass to give a sodium flame. With cæsium alum a constant weight was obtained by keeping the oven at 130° C. to 140° C.

The Observed Densities and Solubilities.

In the following tables the numbers in each column are obtained as follows:—

Column I. gives the temperatures to which the observation recorded in the remaining columns refer. These temperatures are corrected for emergent column, and are, as before stated, the mean temperature corresponding to the meaned density and the meaned solubility.

Column II. gives the density of the saturated solution, obtained as already stated. It summarizes over 600 observations, excluding the 300 mentioned on p. 195.

Column III. gives the corresponding solubilities in parts of anhydrous salt dissolved by 100 parts of water, and is also a summary of some 450 observations exclusive of the above mentioned 300.

Column IV. gives the number of gram-molecules of salt in 1 litre of solution saturated at the temperature recorded in column I. The numbers are obtained by dividing the weight of salt found in the litre by the molecular weight of that salt.

Column V. gives the number of gram-molecules of water in the litre. The numbers are derived by dividing the weight of the water in 1 litre of saturated solution by the molecular weight of water. Throughout this work the atomic weights used are those based on hydrogen as unity and oxygen as equal to 15·88.

Column VI. gives a measure of the concentration—it is the ratio of the number of salt molecules to the sum of salt and water molecules in the same volume of solution.

Column VII. gives the solubilities, taken from COMEY'S 'Dictionary of Solubilities,' of such salts as have already been investigated.

SODIUM Chloride.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.	
° C.							
0·35	1·2090	35·75	5·484	49·81	10·083	35·7	
15·20	1·2020	35·84	5·462	49·49	10·061	35·9	1·2025
30·05	1·1956	36·20	5·473	49·09	9·969	36·3	1·1960
45·40	1·1891	36·60	5·488	48·69	9·872	36·8	1·1895 *
61·70	1·1823	37·28	5·529	48·17	9·712	37·4	1·1827
75·65	1·1764	37·82	5·560	47·74	9·586	38·2	1·1769
90·50	1·1701	38·53	5·606	47·24	9·427	39·1	
Boiling-point 107·0	1·1631	39·65	5·688	46·58	9·189	40·2	

The solubilities were determined by evaporating to dryness in Jena glass bulbs.

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* The numbers in this column are the densities of the saturated solution of NaCl obtained by ANDRIA ('J. Prakt. Chem.,' [2], 30, 305), and reduced to the temperatures given in column I.

POTASSIUM Chloride.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.	
° C.							
0·70	1·1540	28·29	3·438	50·31	15·633	28·7	28·23 *
19·55	1·1738	34·37	4·057	48·84	13·038	34·6	34·06
32·80	1·1839	38·32	4·432	47·87	11·801	38·2	38·05
59·85	1·1980	45·84	5·088	45·95	10·031	45·5	45·47
74·80	1·2032	49·58	5·389	44·99	9·348	49·6	
89·15	1·2069	53·38	5·676	44·01	8·753	53·6	
Boiling-point 108·0	1·2118	58·11	6·018	42·87	8·124	58·5	

The solubilities were determined by evaporating to dryness in platinum crucibles, those at boiling-point in Jena glass bulbs.

RUBIDIUM Chloride.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.
C.						
0·55	1·4409	77·31	5·238	45·44	9·675	76·4 at 1° C.
18·70	1·4865	90·32	5·881	43·69	8·429	82·9 ,, 7 C.
31·50	1·5118	98·61	6·257	42·58	7·805	
44·70	1·5348	106·24	6·590	41·63	7·317	
60·25	1·5558	115·63	6·955	40·35	6·802	
75·15	1·5746	124·52	7·280	39·22	6·387	
89·35	1·5905	132·73	7·562	38·22	6·054	
Boiling-point 114·0	1·6148	146·65	8·003	36·62	5·575	

The solubilities were determined by evaporating to dryness in platinum crucibles, those at boiling-point in Jena glass bulbs.

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* The numbers in this column are the solubilities given by ANDRIA ('J. Prakt. Chem.,' 137, 468) reduced to the temperatures given in column I.

Cæsium Chloride.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.
° C.					
0·70	1·8458	162·29	6·836	39·36	6·758
16·20	1·8984	182·24	7·337	37·62	6·127
29·85	1·9359	197·17	7·688	36·43	5·739
45·55	1·9702	213·45	8·030	35·16	5·379
60·20	2·0012	229·41	8·342	33·98	5·073
76·10	2·0286	245·76	8·630	32·81	4·802
89·50	2·0500	259·56	8·858	31·88	4·599
Boiling-point 119·4	2·0859	289·98	9·283	29·92	4·223

The solubilities were determined in platinum crucibles, except those at the boiling-point, which were done in Jena glass bulbs.

Thallous Chloride.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from Comey's dictionary.
° C.						
0·4	1·0013	0·17	·00707	55·91	7916	0·19
15·6	1·0017	0·29	·01199	55·86	4660	0·27
30·05	·9996	0·47	·01930	55·65	2930	0·40
45·20	·9964	0·72	·02976	55·33	1860	0·52
59·80	·9922	1·03	·04266	54·92	1288	0·74
75·65	·9870	1·48	·06060	54·40	896·6	1·03
89·65	·9821	1·96	·07964	53·87	677·4	1·32
Boiling point 99·35	·9787	2·41	·09684	53·45	552·9	1·55

The solubilities were determined in platinum crucibles, except those at the boiling-point, which were done in Jena glass bulbs.

SODIUM Sulphate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.
° C.						
0·70	1·0432	4·71	·3327	56·05	168·46	5·0
10·25	1·0802	9·21	·6456	55·32	86·63	9·18
15·65	1·1150	14·07	·9747	54·67	57·07	14·12
20·35	1·1546	Lost	—	—	—	—
24·90	1·2067	27·67	1·8534	52·86	29·57	27·6
27·65	1·2459	34·05	2·2425	51·99	24·18	34·1
30·20	1·2894	41·78	2·6926	50·86	19·88	41·8
31·95	1·3230	47·98	3·0400	50·00	17·45	47·6
33·50	1·3307	49·39	3·1174	49·82	16·98	50·5
38·15	1·3229	48·47	3·0608	48·47	17·28	49·3
44·85	1·3136	47·49	2·9980	49·81	17·62	47·7
60·10	1·2918	45·22	2·8507	49·75	18·45	45·3
75·05	1·2728	43·59	2·7383	49·57	19·11	44·0
89·85	1·2571	42·67	2·6643	49·28	19·50	43·1
Boiling-point 101·9	1·2450	42·18	2·6175	48·97	19·71	42·3

Most of the solubilities were determined by evaporating to dryness in platinum crucibles, the remainder in Jena glass bulbs. On plotting out the results of the density and solubility determinations against the temperatures, it will be seen that both curves give the transition point at 32°·5. The direct estimation of the melting-point of hydrated sodium sulphate made by Messrs. RICHARDS and CHURCHILL* gave 32°·379 on the hydrogen scale.

POTASSIUM Sulphate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.		
° C.								
0·40	1·0589	7·47	·4253	55·11	130·66	8·5	7·42	
15·70	1·0770	10·37	·5849	54·58	94·31	10·4	10·31	
31·45	1·0921	13·34	·7429	53·89	73·52	12·5	13·22	†
42·75	1·1010	15·51	·8551	53·31	63·35	14·5	15·25	
58·95	1·1086	18·01	·9792	52·53	54·65	17·6	17·99	
74·85	1·1157	20·64	1·1036	51·72	47·86	20·8		
89·70	1·1194	22·80	1·2019	50·98	43·41	23·8		
Boiling-point 101·1	1·1207	24·21	1·2621	50·47	40·99	26·4		

The solubilities were determined by evaporating in platinum crucibles, those at the boiling-point in Jena glass bulbs.

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* 'Zeit. Phys. Chem.,' 26, 690 (1898).

† Reduced to the temperatures in column I., from ANDRIA'S solubilities, see 'J. Prakt. Chem.,' 137, 471.

Rubidium Sulphate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from Comey's dictionary.
° C.						
0·50	1·2740	36·66	1·2903	52·14	41·41	36·3
15·80	1·3287	46·04	1·5810	50·89	33·19	45·3
31·60	1·3704	54·25	1·8193	49·69	28·32	55·9
44·20	1·3998	60·75	1·9970	48·70	25·35	65·5
57·90	1·4232	66·59	2·1475	47·78	23·24	71·2
74·75	1·4480	73·25	2·3111	46·74	21·22	74·5
89·45	1·4649	78·61	2·4337	45·87	19·84	77·5
Boiling-point 102·4	1·4753	82·57	2·5185	45·19	18·94	80·2

The solubilities were determined by evaporating to dryness in platinum crucibles those at the boiling-point in Jena glass bulbs.

Cæsium Sulphate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from Comey's dictionary.
° C.						
0·70	1·9766	167·55	3·4467	41·32	12·99	158·7 at −2° C.
15·00	1·9992	176·02	3·5499	40·51	12·41	—
30·40	2·0202	184·35	3·6469	39·74	11·90	—
44·90	2·0365	192·49	3·7318	38·94	11·44	—
59·50	2·0512	199·35	3·8035	38·32	11·08	—
75·70	2·0664	207·89	3·8850	37·54	10·66	—
89·75	2·0774	214·82	3·9471	36·91	10·35	—
Boiling-point 108·6	2·0932	224·50	4·0323	36·08	9·95	—

The solubilities were determined by evaporating to dryness in the Jena glass bulbs.

THALLOUS Sulphate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration	VII. Solubility, from COMEY'S dictionary.
° C.						
0·15	1·0248	2·72	·0541	55·80	1032·5	2·8
15·60	1·0384	4·32	·0858	55·67	649·9	4·6
29·80	1·0512	6·13	·1214	55·40	457·3	6·7
44·95	1·0652	8·39	·1647	54·97	334·7	8·8
60·40	1·0795	10·96	·2130	54·41	256·4	11·2
75·90	1·0941	13·84	·2654	53·76	203·5	13·8
90·05	1·1071	16·54	·3138	53·14	170·3	16·8
Boiling-point 99·7	1·1165	18·45	·3474	52·72	152·7	18·7

The solubilities were determined by evaporating to dryness in the Jena glass bulbs.

SODIUM Nitrate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.
° C.						
0·30	1·3530	73·30	6·776	43·660	7·443	73·4
15·45	1·3769	84·48	7·466	41·74	6·591	85·7
30·00	1·3992	96·15	8·121	39·89	5·912	95·0
44·50	1·4210	109·10	8·779	38·01	5·329	106·5
60·00	1·4446	124·56	9·489	35·98	4·792	124·3
76·15	1·4701	143·15	10·248	33·81	4·300	142·1
90·25	1·4920	161·61	10·917	31·88	3·920	163·5
Boiling-point 119·0	1·5374	208·84	12·310	27·84	3·262	c̄ 220

The solubilities were determined by evaporating to dryness in Jena glass bulbs; and it is to be noted that a small trace of salt was always found in the distillate from the bulbs.

POTASSIUM Nitrate

I. Temperature	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI Con-centration.	VII. Solubility, from COMEY'S dictionary.		
°C.								
0·40	1·0817	13·43	1·276	53·34	42·80	13·5	13·53	*
14·90	1·1389	25·78	2·326	50·64	22·77	25·9	25·75	
30·80	1·2218	47·52	3·921	46·32	12·81	45·7	47·28	
44·75	1·3043	74·50	5·547	41·80	8·536	73·5	73·67	
60·05	1·3903	111·18	7·291	36·83	6·051	111·1	110·02	
76·00	1·4700	156·61	8·936	32·04	4·585	159·0	—	
91·65	1·5394	210·20	10·391	27·75	3·767	212·6	—	
Boiling-point 114·0	1·6269	311·64	12·269	22·10	2·801	327·1	—	

The solubilities were determined in platinum crucibles, those at the boiling-point in Jena glass bulbs.

RUBIDIUM Nitrate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Con-centration.	VII. Solubility, from COMEY'S dictionary.
°C.						
0·60	1·1389	20·39	1·318	52·91	42·66	20·1 at 0° C.
15·85	1·2665	44·28	2·656	49·09	19·49	43·5 ,, 10° C.
31·55	1·4483	86·67	4·592	43·42	10·45	—
45·85	1·6216	139·38	6·450	37·90	6·875	—
63·40	1·8006	217·06	8·423	31·76	4·770	—
75·60	1·9055	284·06	9·630	27·75	3·881	—
90·95	2·0178	382·89	10·932	23·37	3·138	—
Boiling-point 118·3	2·1867	617·11	12·858	17·05	2·326	—

The solubilities were determined by evaporating to dryness in platinum crucibles, except in the case of the observations at the boiling-point; these were done in Jena glass bulbs, and the distillate always showed a trace of nitrate as having come over.

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* These solubilities are calculated from those of ANDRIA at slightly different temperatures, 'J. Prakt. Chem.,' 137, 474.

CÆSIUM Nitrate

I Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.
° C.						
0·35	1·0701	9·54	·4815	54·64	114·35	10·58 at 3°·2 C.
15·95	1·1345	19·46	·9555	53·11	56·56	—
30·45	1·2219	34·50	1·6205	50·81	32·36	—
45·15	1·3306	55·58	2·4572	47·84	20·47	—
59·90	1·4565	83·50	3·4260	44·39	13·96	—
76·40	1·6068	124·64	4·6082	40·00	9·68	—
90·55	1·7307	165·18	5·5724	36·50	7·55	—
Boiling-point 106·2	1·8657	220·32	6·6351	32·56	5·91	—

The solubilities were determined by evaporating to dryness in platinum crucibles; those at the boiling-point in Jena glass bulbs.

THALLOUS Nitrate.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.
° C.						
0·65	1·0346	4·07	·1532	55·60	363·92	—
15·40	1·0653	7·93	·2963	55·20	187·50	9·7 at 18° C.
30·60	1·1150	14·63	·5385	54·40	102·06	—
44·65	1·1891	24·98	·8995	53·22	60·13	—
57·30	1·2986	41·31	1·4369	51·40	36·77	43·5 at 58° C.
64·95	1·3957	56·33	1·9036	49·93	27·33	—
76·00	1·6096	91·93	2·9183	46·90	17·07	—
87·80	2·0258	174·02	4·8780	41·23	9·45	—
Boiling-point 104·5	3·1906	593·93	10·3366	25·72	3·488	588·2 at 107° C.

The solubilities were determined by evaporating to dryness in platinum crucibles; those at the boiling-point in Jena glass bulbs.

POTASSIUM Alum

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.
° C.						
0·40	1·0292	3·01	·0586	55·88	954·6	3·05
15·30	1·0461	5·09	·0989	55·67	564·0	5·06
28·10	1·0661	7·83	·1510	55·28	367·1	7·50
43·20	1·1044	13·31	·2530	54·51	216·5	13·40
60·45	1·1835	25·06	·4624	52·93	115·5	25·7

The solubilities were determined in the Jena glass bulbs. Column IV. gives the number of grain molecules in the litre calculated on the assumption that the anhydrous salt is $K_2Al_2\ (SO_4)_4$.

RUBIDIUM Alum.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from COMEY'S dictionary.
0° C.						
0·40	1·0072	0·73	·0121	55·92	4607	0·73
15·20	1·0112	1·28	·0211	55·84	2645	1·33
32·20	1·0165	2·38	·0391	55·53	1420	2·44
45·80	1·0267	4·13	·0673	55·14	820·6	4·33
59·65	1·0466	7·27	·1241	54·57	440·8	7·97
69·75	1·0804	12·23	·1947	53·84	277·5	13·42

Solubilities in Jena glass bulbs. Column IV. gives the number of gram-molecules calculated on assumption that the anhydrous salt is $Rb_2Al_2\ (SO_4)_4$.

Cæsium Alum.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.	VII. Solubility, from Comey's dictionary.
0° C.						
0·40	1·0017	0·21	·0030	55·91	18890	0·19
15·60	1·0022	0·35	·0050	55·85	11230	0·36
29·15	1·0010	0·58	·0082	55·66	6773	0·59
45·25	·9994	1·07	·0151	55·30	3658	1·07
60·60	1·0004	2·05	·0287	54·83	1911	2·04
75·35	1·0107	4·32	·0599	54·19	905·6	4·39
83·05	1 0250	6·86	·0934	53 67	575·6	5·87
90·85	1·0328	11·26	·1524	52·92	353·8	—
Boiling-point 100·4	1·1285	22·84	·3002	51·38	172·2	—

Solubilities in Jena glass bulb. Numbers in IVth column are derived from assumption that anhydrous salt is CS_2, $Al_2 (SO_4)_4$.

Thallium Alum.

I. Temperature.	II. Density.	III. Solubility.	IV. Number of gram-molecules of salt.	V. Number of gram-molecules of aqua.	VI. Concentration.
0° C.					
0·45	1·0299	3·22	·0382	55·81	1460
16·10	1·0503	5·61	·0664	55·62	839·3
29·85	1·0808	9·33	·1097	55·29	505·0
37·50	1·1090	13·09	·1527	54·85	360·2
45·20	1·1500	18·50	·2136	54·28	255·1
52·40	1·2051	25·39	·2904	53·75	186·1
60·05	1·2812	35·43	·3988	52·91	133·7

Solubilities in platinum crucibles. Numbers in IVth column are calculated for $N_2Al_2 (SO_4)_4$.

In conclusion, I am glad to have this opportunity of thanking Mr. E. G. Hartley for his help in the observations on the densities at the boiling-points, and Messrs. Neville and Whetham for the kind interest they have taken in the work, and for several suggestions.

mic *Problem.*

4.

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IX. *The Third Elliptic Integral and the Ellipsotomic Problem.*

By A. G. GREENHILL, *F.R.S.*

Received December 22, 1903,—Read January 21, 1904.

THE ABEL Centennial Ceremony, held in Christiania, September, 1902, has directed the attention of mathematicians to the great influence of ABEL on modern analysis, and to the history of elliptic functions, and of the foundation by CRELLE of the "Journal für die reine und angewandte Mathematik."

ABEL's article in the first volume of 'CRELLE's Journal,' 1826.

"Ueber die Integration der Differential-Formel

$$\frac{\rho\, dx}{\sqrt{R}} \qquad (A),$$

wenn R und ρ ganze Functionen sind," is of great importance as indicating the existence of what is now called the *pseudo-elliptic integral*; the present memoir is intended to show the utility of this integral in its application to mechanical theory.

Provided with a list of these integrals, proceeding from the simplest case and extending as far as possible, the student of applied mathematics will be able to effect the complete solution of many interesting mechanical problems now abandoned in an unfinished state; at the same time, the exploration along the simplest lines of progress is effected of the general analytical field, and mathematical research is guided in a path likely to arrive at useful development in the theory of elliptic functions.

The problem of the Division, and thence also of the Transformation, of elliptic functions is solved incidentally; as also the difficulties raised by LEGENDRE in his letter to ABEL, of January 16, 1829:—

> "Mais de là naît une difficulté sur la division générale des fonctions elliptiques.
>
> "Admettrons-nous cette différence énorme entre les fonctions de 3^{me} espèce à paramètre logarithmique et les fonctions à paramètre circulaire, savoir que les premières peuvent s'exprimer par une fonction de deux variables, facilement reductible en table, et que les autres ne le peuvent pas? Il y aurait donc par le fait quatre espèces de fonctions elliptiques au lieu de trois, et la quatrième serait bien plus composée que la troisième. C'est un point qui mérite d'être examiné et mis au clair. Je le recommande à votre investigation et à celle de M. JACOBI."

But a mere change of sign in ABEL's results is sufficient to pass from the logarithmic to the circular form of the third elliptic integral, and as it is the circular form which is of almost invariable occurrence in dynamical problems, we shall adopt it as our standard form.

Applications of this third elliptic integral are introduced in the course of the memoir to these problems, such as POINSOT's herpolhode, and the spinning top or gyroscope, the spherical catenary, the velarium, and the elastica under uniform normal pressure.

References to former articles on the same subject in the 'Proceedings of the London Mathematical Society,' vols. 25 and 27, are given in the sequel in the abbreviated form, L.M.S., 25 or 27; frequent reference is made also to KIEPERT's articles on the theory of elliptic functions in the 'Mathematische Annalen' ('Math. Ann.,' vols. 26, 32, 37).

1. Working then with the third elliptic integral in the circular form, when the elliptic parameter v is a fraction of the imaginary period ω_3, we change the variable in the standard form of WEIERSTRASS,

$$\int_0^u \frac{\frac{1}{2} i\wp' v \, du}{\wp u - \wp v} = i \log e^{v\zeta v} \sqrt{\frac{\sigma(u-v)}{\sigma(u+v)}} \qquad (1),$$

by putting

$$\wp u - \wp v = s - \sigma, \quad \wp'^2 u = \mathrm{S} \qquad (2),$$

$$i\wp' v = \sqrt{-\Sigma}, \quad v = f\omega_3 \qquad (3),$$

where f is a real fraction, so that the integral changes to

$$\int_s^\infty \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma} \cdot \frac{ds}{\sqrt{\mathrm{S}}} = i \log e^{v\left(\zeta v - \frac{\eta}{\omega} v\right)} \sqrt{\frac{\theta(u-v)}{\theta(u+v)}} \qquad (4)$$

(where the elliptic arguments u and v may be supposed for a moment to be normalised to the Jacobian form), and s is an elliptic function of u which we may denote by $s(u)$, differing from WEIERSTRASS's $\wp u$ by a constant, so that

$$s'(u) = \wp' u, \quad s''(u) = \wp'' u \ldots,$$

while

$$\sigma = s(v), \quad is'(v) = \sqrt{-\Sigma}.$$

Considering that (HALPHEN, 'Fonctions Elliptiques,' 1, p. 222)

$$\frac{\theta(u-v)}{\theta(u+v)} \qquad (5)$$

is an algebraical function of s when v is an aliquot part of a period, we take

$$i \log \sqrt{\frac{\theta(u+v)}{\theta(u-v)}} = \int_s^\infty \frac{\mathrm{P}(v)(s-\sigma) - \frac{1}{2}\sqrt{-\Sigma}}{s-\sigma} \frac{ds}{\sqrt{\mathrm{S}}} \qquad (6),$$

where
$$i\mathrm{P}(v) = \frac{\eta}{\omega} v - \zeta v \qquad (7),$$
and denote this integral by I (v), and work with it as our standard form of the elliptic integral of the III. kind.

The function employed by HALPHEN ('F E.,' I, p. 230) is now
$$\phi(u, \quad v) = \sqrt{(s-\sigma)} \exp\{\quad iu\mathrm{P}(v) - i\mathrm{I}(v)\} \qquad (8),$$
$$\phi(u, -v) = \sqrt{(s-\sigma)} \exp\{-iu\mathrm{P}(v) + i\mathrm{I}(v)\} \qquad (9),$$
and is a Lamé function of the first order, satisfying his differential equation
$$\frac{1}{\phi}\frac{d^2\phi}{du^2} - 2\wp u - \wp v = 0 \qquad (10);$$
thence the Lamé functions of higher order may be derived by differentiation, as shown by HERMITE, 'Comptes Rendus,' 1877, and these can be employed in the problems considered by Professor G. H. DARWIN, in the 'Phil. Trans.,' 197, 1901, "Ellipsoidal Harmonic Analysis."

In the Hermite-Jacobi notation we may take
$$\phi(u, v) = \frac{\theta 0 \theta(u - v)}{\theta u \theta v} \exp(u\,\mathrm{zs}v), \quad \text{or} \quad \frac{\theta 0 \mathrm{H}(u - v)}{\theta u \mathrm{H} v} \exp(u\,\mathrm{zn}v) \qquad (11).$$

2. Next introduce the x and y employed by HALPHEN ('F.E.,' I, p. 103), which may be connected with the a, b, and p of ABEL's notation ('Œuvres,' II., p. 155) by
$$x = -\frac{16b^3}{p^2}, \quad y = -1 - \frac{4ab}{p} \qquad (1),$$
and put
$$s'^2(u) = \mathrm{S} = 4s(s+x)^2 - \{(1+y)s + xy\}^2 \qquad (2),$$
$$s(u) = \wp u + \frac{(1+y)^2 - 8x}{12} \qquad (3);$$
then if we put
$$s - \sigma = s(u) - s(v) = s + x \qquad (4),$$
$$\sigma = s(v) = -x, \quad \wp(v) = -\frac{(1+y)^2 + 4x}{12} \qquad (5),$$
$$is'(v) = i\wp'(v) = \sqrt{-\Sigma} = x \qquad (6).$$

The multiplicative values
$$s(2v), \quad s(3v), \ldots\ s(nv) \qquad (7),$$
$$is'(2v), \quad is'(3v), \ldots\ is'(nv) \qquad (8),$$
can now all be expressed rationally in terms of x and y, with the γ functions of HALPHEN ('F.E.,' I, p. 102; also ABEL, 'Œuvres,' II., p. 159; MONTARD-PONCELOT, 'Applications d'analyse et de géométrie,' t. I., Paris, 1862), by means of the recurring formulas
$$s(mv) - s(nv) = x^{\frac{1}{3}} \frac{\gamma_{m+n}\gamma_{m-n}}{\gamma_m^2 \gamma_n^2}, \quad is'(nv) = x \frac{\gamma_{2n}}{\gamma_n^4} \qquad (9).$$

Thus

$$s(v) = -x, \qquad is'(v) = x \qquad (10),$$

$$s(2v) = 0, \qquad is'(2v) = xy \qquad (11),$$

$$s(3v) = y - x, \qquad is'(3v) = y - x - y^2 \qquad (12),$$

$$s(4v) = \frac{x(y-x)}{y^2} - x, \qquad is'(4v) = x\frac{x(y-x-y^2)-(y-x)^2}{y^3} \qquad (13),$$

$$s(5v) = \frac{xy(y-x-y^2)}{(y-x)^2} - x, \quad is'(5v) = x\frac{y^2(xy-x^2-y^3)-x(y-x-y^2)^2}{(y-x)^3} \qquad (14),$$

$$s(6v) = \frac{(y-x)(xy-x^2-y^3)}{(y-x-y^2)^2} - x,$$

$$is'(6v) = y\frac{(y-x)^2\{x(y-x-y^2)-(y-x)^2\}-(xy-x^2-y^3)}{(y-x-y^2)^3} \qquad (15),$$

$$= x^3\frac{\gamma_{n+2}\gamma_{n-2}}{\gamma_n^{\;2}},$$

$$s(nv) = x^2\frac{\gamma_{n+1}\gamma_{n-1}}{\gamma_n^{\;2}} - x, \quad is'(nv) = x\frac{\gamma_{2n}}{\gamma_n^{\;4}} \qquad (16).$$

3. For the determination of P (nv) we have the formula (Halphen, 'F.E.,' 1, p. 102)

$$\frac{\sigma(nv)}{(\sigma v)^{n^2}} = \psi_n(v) = (\wp' v)^{\frac{n^2-1}{3}}\gamma_n \qquad (1);$$

whence, by logarithmic differentiation,

$$n\mathrm{P}(v) - \mathrm{P}(nv) = i(n\zeta v - \zeta nv)$$

$$= \frac{n^2-1}{3n}\frac{\wp'' v}{i\wp' v} + \frac{1}{n\gamma_n}\frac{d\gamma_n}{dvi}$$

$$= \frac{n^2-1}{3n}(1+y) + \frac{1}{n\gamma_n}\left(\frac{\delta\gamma_n}{\delta x}\frac{dx}{dvi} + \frac{\delta\gamma_n}{\delta y}\frac{dy}{dvi}\right) \qquad (2).$$

Now for every homogeneity factor, and as elliptic functions of degree zero,

$$y = \frac{\wp' 2v}{\wp' v} = \frac{s' 2v}{s' v} \qquad (3),$$

$$\frac{1}{y}\frac{dy}{dvi} = 2\frac{\wp'' 2v}{i\wp' 2v} - \frac{\wp'' v}{i\wp' v}$$

$$= 2\frac{2x^2 - xy(1+y)}{xy} - \frac{x(1+y)}{x}$$

$$= 4\frac{x}{y} - 3(1+y) \qquad (4),$$

and
$$x = \frac{(\wp 2v - \wp v)^3}{\wp'^2 v} \qquad (5),$$

$$\frac{1}{x}\frac{dx}{dv} = -3\,\frac{2i\wp' 2v - i\wp' v}{\wp 2v - \wp v} - 2\,\frac{\wp'' v}{i\wp' v}$$

$$= -3\,\frac{2xy - x}{x} - 2(1+y)$$

$$= 1 - 8y \qquad (6),$$

so that
$$nP(v) - P(nv) = \frac{n^2-1}{3n}(1+y)$$
$$+ \frac{1}{n\gamma_n}\left\{x(1-8y)\frac{\delta\gamma_n}{\delta x} + (4x - 3y - 3y^2)\frac{\delta\gamma_n}{\delta y}\right\} \qquad (7).$$

Thus, putting $n = 2, 3, 4, \ldots,$

$$2P(v) - P(2v) = \tfrac{1}{2}(1+y) \qquad (8),$$

$$3P(v) - P(3v) = 1 \qquad (9),$$

$$4P(v) - P(4v) = \tfrac{1}{2}(1+y) + \frac{x}{y} \qquad (10).$$

$$5P(v) - P(5v) = 1 + \frac{y^2}{y-x} \qquad (11),$$

$$6P(v) - P(6v) = \tfrac{1}{2}(1+y) + 1 + \frac{y(y-x)}{y-x-y^2} \qquad (12),$$

$$7P(v) - P(7v) = 2 + y + \frac{y(y-x)^2}{\gamma_7} \qquad (13),$$

$$8P(v) - P(8v) = \tfrac{1}{2}(1+y) + 2 + \frac{(y-x)^2(y-x-y^2)}{\gamma_8} \qquad (14),$$

$$9P(v) - P(9v) = 3 + \frac{x^4(y-x-y^2)^3}{\gamma_9} \qquad (15),$$

$$10P(v) - P(10v) = \tfrac{1}{2}(1+y) + \frac{N_{10}}{\gamma_{10}} \qquad (16),$$

$$N_{10} = 2x^4(1+y) + x^3(-6y + y^2 + y^3) + x^2(6y^2 - 6y^3 - y^4)$$
$$+ x(-2y^3 + 4y^4 + y^5) - y^6 - y^7.$$

$$11P(v) - P(11v) = \frac{N_{11}}{\gamma_{11}} \qquad (17),$$

$$N_{11} = 3x^5 + 3x^4(-3y + y^2) + x^3(9y^2 + 2y^3 + 8y^4)$$
$$+ x^2(-3y^3 - 15y^4 - 9y^5 + 5y^6) + x(12y^5 + 3y^6 - 2y^7 + y^8) - 2(y^6 + y^7).$$

4. When v is an aliquot μ^{th} part of the imaginary period $2\omega_3$, so that

$$v = \frac{2\omega_3}{\mu}, \quad \text{or} \quad \frac{2r\omega_3}{\mu} \qquad (1).$$

and μv is congruent to a period, then

$$\mathrm{P}(\mu v) = \infty, \quad \mathrm{P}(mv) + \mathrm{P}(\mu - m)v = 0 \qquad (2),$$

so that $\mu \mathrm{P}(v)$ is given by equation (7), § 3, by putting $n = \mu - 1$, with the additional condition that

$$s(\mu v) = \infty, \quad s(mv) - s(\mu - m)v = 0 \qquad (3),$$

or

$$\gamma_\mu = 0 \qquad (4).$$

Thus

$$\gamma_3 = 0, \quad 3\mathrm{P}(v) = \tfrac{1}{2}(1 + y),$$

$$\gamma_4 = 0, \quad 4\mathrm{P}(v) = 1,$$

$$\gamma_5 = 0, \quad 5\mathrm{P}(v) = \tfrac{1}{2}(3 + x),$$

$$\gamma_6 = 0, \quad 6\mathrm{P}(v) = 2,$$

$$\gamma_7 = 0, \quad 7\mathrm{P}(v) = \tfrac{1}{2}(1 + y) + 1 + \frac{x}{y},$$

$$\gamma_8 = 0, \quad 8\mathrm{P}(v) = 2 + y + \frac{xy}{y - x},$$

$$\gamma_9 = 0, \quad 9\mathrm{P}(v) = \tfrac{1}{2}(1 + y) + 1 + \frac{x}{y} + \frac{y^2}{y - x},$$

$$\gamma_{10} = 0, \quad 10\mathrm{P}(v) = 2 + y + \frac{x}{y} + \frac{y(y - x)}{y - x - y^2},$$

$\gamma_{11} = 0$, $\mathrm{P}(5v) + \mathrm{P}(6v) = 0$, so that adding (11) and (12), § 3,

$$11\mathrm{P}(v) = 2 + \tfrac{1}{2}(1 + y) + \frac{y^2}{y - x} + \frac{y(y - x)}{y - x - y^2},$$

and, similarly,

$$\gamma_{13} = 0, \quad 13\mathrm{P}(v) = 3 + y + \tfrac{1}{2}(1 + y) + \frac{y(y - x)}{y - x - y^2} + \frac{y(y - x)^2}{\gamma_7},$$

$$\gamma_{15} = 0, \quad 15\mathrm{P}(v) = 4 + y + \tfrac{1}{2}(1 + y) + \frac{y(y - x)^2}{\gamma_7} + \frac{(y - x)^2(y - x - y^2)}{\gamma_8},$$

$$\gamma_{17} = 0, \quad 17\mathrm{P}(v) = 5 + \tfrac{1}{2}(1 + y) + \frac{(y - x)^2(y - x - y^2)}{\gamma_8} + \frac{(y - x - y^2)^3}{x^{-1}\gamma_9},$$

$$\gamma_{19} = 0, \quad 19\mathrm{P}(v) = 3 + \tfrac{1}{2}(1 + y) + \frac{(y - x - y^2)^3}{x^{-1}\gamma_9} + \frac{\mathrm{N}_{10}}{\gamma_{10}},$$

$$\gamma_{21} = 0. \quad 21\mathrm{P}(v) = \tfrac{1}{2}(1 + y) + \frac{\mathrm{N}_{10}}{\gamma_{10}} + \frac{\mathrm{N}_{11}}{\gamma_{11}}.$$

The ellipsotomic problem of the determination of the division-values (*Theilwerthe*) of the elliptic functions resolves itself thus into a consideration of the curve in x and y given by this rational equation (4), which may thus be called the *ellipsotomic equation*, by analogy with the *cyclotomic equation* for the circular functions; and the problem may be considered solved when x and y can be expressed, rationally or irrationally, in terms of a parameter; according to POINCARÉ (' Bulletin de la Société Mathématique de France,' 1883, 11, p. 112) this can always be effected by uniform functions of an independent variable.

The details have been carried out in the ' Proceedings of the London Mathematical Society' (L.M.S.), 25, for values of μ up to 22 inclusive, omitting 19, which still remains awaiting solution.

When μ is an even number, $4n + 2$ or $4n$, $s\left(\frac{1}{2}\mu v\right) = s\left(r\omega_3\right)$ is a root of $S = 0$, so that the cubic S can be resolved into factors, and we can employ the functions of the Second Stage of LEGENDRE and JACOBI, as required in most dynamical applications.

But when μ is odd, this resolution cannot be effected algebraically, and WEIERSTRASS's functions of the First Stage must be employed.

The ellipsotomic relation (L.M.S., 27, p. 405)

$$\gamma_\mu = 0 \qquad \text{and} \qquad \frac{\gamma\left(\mu - n\right)}{\gamma\left(n\right)} = \lambda^{\mu-2n} \tag{5}$$

are equivalent, so that

$$\mu \text{ odd,} \quad \lambda = \frac{\gamma\,\frac{1}{2}\left(\mu + 1\right)}{\gamma\,\frac{1}{2}\left(\mu - 1\right)}, \qquad \lambda^3 = \frac{\gamma\,\frac{1}{2}\left(\mu + 3\right)}{\gamma\,\frac{1}{2}\left(\mu - 3\right)}, \ldots \qquad (6),$$

$$\mu \text{ even,} \quad \lambda^2 = \frac{\gamma\,\frac{1}{2}\left(\mu + 2\right)}{\gamma\,\frac{1}{2}\left(\mu - 2\right)}, \qquad \lambda^4 = \frac{\gamma\,\frac{1}{2}\left(\mu + 4\right)}{\gamma\,\frac{1}{2}\left(\mu - 4\right)}, \ldots \qquad (7).$$

Writing v for $\dfrac{2\omega_3}{\mu}$, and ω for ω_3,

$$\frac{\sigma n v}{\left(\sigma v\right)^{n^2}} = \psi_n\left(v\right) = x^{\frac{1}{4}\left(n^2-1\right)}\gamma_n \tag{8}$$

(HALPHEN, ' F.E.,' 1, pp. 102, 198); and changing n into $\mu - n$, and dividing,

$$\frac{\sigma\left(\mu - n\right)v}{\sigma n v} = \left[\sigma v x^{\frac{1}{4}}\lambda^{\frac{1}{\mu}}\right]^{\mu^2-2n\mu} \qquad (9).$$

But

$$\sigma\left(\mu - n\right)v = \sigma\left(2\omega - nv\right) = e^{\eta v\left(\mu-2n\right)}\sigma n v \qquad (10),$$

so that

$$e^{-\frac{\eta v}{\mu}}\sigma v = x^{-\frac{1}{4}}\lambda^{-\frac{1}{\mu}} \qquad (11),$$

and generally

$$e^{-\frac{n^2\eta v}{\mu}}\sigma n v = x^{\frac{1}{4}}\lambda^{-\frac{n}{\mu}}\gamma_n \qquad (12),$$

and this is KIEPERT's function $\tau\left(\dfrac{2n\omega}{\mu}\right)$, defined in ' Math. Ann.,' 32, p. 6.

According to HALPHEN ('Math. Ann.,' 15, p. 359; 'Comptes Rendus,' March, 1879)

$$\gamma_n = \frac{\mathrm{H}(nv)\,\mathrm{H}(v)^{\frac{1}{3}(n^2-4)}}{\mathrm{H}(2v)^{\frac{1}{3}(n^2-1)}} \qquad (13).$$

5. The integral employed by ABEL, changed to the circular form, can be written

$$\mathrm{J}(v) = \int \frac{\mathrm{A}z + \mathrm{K}}{\sqrt{\mathrm{Z}}}\,dz \qquad (1),$$

$$\mathrm{Z} = z - (\mathrm{A}z^2 + \mathrm{B}z + \mathrm{C})^2 \qquad (2),$$

and, as pointed out in the 'Archiv der Mathematik und Physik,' 3. Reihe, I., p. 72, this is exactly in the form required in the problem of LÉVY's 'Elastica,' with $z = r^2/a^2$.

We reduce it to our standard form in (4), § 1, by putting

$$z = \frac{s(u) - s(3v)}{s(u) - s(v)} = 1 - \frac{y}{s + x} \qquad (3),$$

$$\mathrm{Z} = z - (\mathrm{A}z^2 + \mathrm{B}z + \mathrm{C})^2 = \frac{\frac{1}{4}\mathrm{S}}{(s + x)^4} \qquad (4),$$

so that

$$\begin{aligned}(\mathrm{A}z^2 + \mathrm{B}z + \mathrm{C})^2 &= 1 - \frac{y}{s + x} - \frac{\frac{1}{4}\mathrm{S}}{(s + x)^4} \\ &= \frac{\{2(s + x)^2 - (1 + y)(s + x) + x\}^2}{4(s + x)^4} \qquad (5),\end{aligned}$$

$$\mathrm{A}z^2 + \mathrm{B}z + \mathrm{C} = 1 - \frac{1 + y}{2(s + x)} + \frac{x}{2(s + x)^2} \qquad (6),$$

and therefore

$$\mathrm{A} = \frac{x}{2y^2}, \qquad \mathrm{B} = \frac{-2x + y + y^2}{2y^2}, \qquad \mathrm{C} = \frac{x - y + y^2}{2y^2} \qquad (7),$$

$$\mathrm{A} + \mathrm{B} + \mathrm{C} = 1 \qquad (8),$$

and $z - 1$ is a factor of Z corresponding to $s = \infty$.

More generally, with

$$\mathrm{M}z = \frac{s(u) - s(3nv)}{s(u) - s(nv)} \qquad (9),$$

so that $1/\mathrm{M}$ is the value of z corresponding to $s = \infty$; then in ABEL's notation, changing the sign of z and Z to obtain the circular form,

$$\mathrm{Z} = pz - (z^2 - az + b)^2 = \frac{\frac{1}{4}\mathrm{QS}}{\{s - s(nv)\}^4} \qquad (10),$$

$$(z^2 - az + b)^2 = \frac{p}{\mathrm{M}}\,\frac{s - s(3nv)}{s - s(nv)} - \frac{\mathrm{QS}}{4\{s - s(nv)\}^4} = \frac{\mathrm{N}}{4\mathrm{M}t^4} \qquad (11),$$

on putting

$$s - s(nv) = t \qquad (12),$$

and

$$\begin{aligned} \mathrm{N} &= 4pt^3\{t - s(3nv) + s(nv)\} \\ &\quad - 4\mathrm{MQ}\{t + s(nv)\}\{t + s(nv) + x\}^2 \\ &\quad + \mathrm{MQ}\{(1+y)t + (1+y)s(nv) + xy\}^2 \\ &= 4pt^4 - 4p\left\{s(3nv) - s(nv) + \frac{\mathrm{MQ}}{p}\right\}t^3 \\ &\quad + \mathrm{MQ}\{(1+y)^2 - 8x - 12s(nv)\}t^2 - 2\mathrm{MQ}s''(nv) - \mathrm{MQ}s'^2(nv) \qquad (13), \end{aligned}$$

which is a perfect square in the form

$$\begin{aligned} \mathrm{N} &= \left[2\sqrt{p}t^2 - \sqrt{p}\left\{s(3nv) - s(nv) + \frac{\mathrm{MQ}}{p}\right\}t + \sqrt{\mathrm{MQ}}is'(nv)\right]^2 \\ &= \left\{2\sqrt{p}t^2 - \sqrt{\mathrm{MQ}}\frac{s''(nv)}{is'(nv)} + \sqrt{\mathrm{MQ}}is'(nv)\right\}^2 \qquad (14), \end{aligned}$$

implying that

$$s(3nv) - s(nv) + \frac{\mathrm{MQ}}{p} = \sqrt{\frac{\mathrm{MQ}}{p}}\frac{s''(nv)}{is'(nv)} \qquad (15),$$

$$\begin{aligned} \frac{\mathrm{MQ}}{p}\{(1+y)^2 - 8x - 12s(nv)\} &= \frac{\mathrm{MQ}}{p}\left\{\frac{s''(nv)}{is'(nv)}\right\}^2 + 4\sqrt{\frac{\mathrm{MQ}}{p}}is'(nv) \\ &= \frac{\mathrm{MQ}}{p}\{(1+y)^2 - 8x - 4s(2nv) - 8s(nv)\} + 4\sqrt{\frac{\mathrm{MQ}}{p}}is'(nv) \qquad (16). \end{aligned}$$

$$\sqrt{\frac{\mathrm{MQ}}{p}} = \frac{is'(nv)}{s(2nv) - s(nv)} \qquad (17),$$

which also satisfies (16).

Then

$$\mathrm{M}z = 1 - \frac{s(3nv) - s(nv)}{t} \qquad (18),$$

and

$$\begin{aligned} z^2 - az + b &= \frac{2\sqrt{p}t^2 - \sqrt{\mathrm{MQ}}\frac{s''(nv)}{is'(nv)}t + \sqrt{\mathrm{MQ}}is'(nv)}{2\sqrt{\mathrm{M}}t^2} \\ &= \sqrt{\frac{p}{\mathrm{M}}} - \tfrac{1}{2}\sqrt{\mathrm{Q}}\frac{s''(nv)}{is'(nv)}\frac{1 - \mathrm{M}z}{s(3nv - s(nv)} + \tfrac{1}{2}\sqrt{\mathrm{Q}}is'(nv)\left\{\frac{1 - \mathrm{M}z}{s(3nv) - s(nv)}\right\}^2 \qquad (19), \end{aligned}$$

requiring the relations

$$1 = \tfrac{1}{2}\sqrt{\mathrm{Q}}is'(nv)\frac{\mathrm{M}^2}{\{s(3nv) - s(nv)\}^2} \qquad (20),$$

$$\begin{aligned} a &= \tfrac{1}{2}\mathrm{M}\sqrt{\mathrm{Q}}\left[\frac{s''(nv)}{is'(nv)}\frac{1}{s(3nv) - s(nv)} + \frac{2is'(nv)}{\{s(3nv) - s(nv)\}^2}\right] \\ &= \tfrac{1}{2}\mathrm{M}\sqrt{\mathrm{Q}}\frac{s''(2nv)}{is'(2nv)}\frac{1}{s(3nv) - s(nv)} \qquad (21), \end{aligned}$$

$$b = \sqrt{\frac{p}{\mathrm{M}}} - \tfrac{1}{2}\sqrt{\mathrm{Q}}\,\frac{s''(nv)}{is'(nv)}\,\frac{1}{s(3nv) - s(nv)} + \tfrac{1}{2}\sqrt{\mathrm{Q}}\,\frac{is'(nv)}{\{s(3nv) - s(nv)\}^2}$$

$$= \sqrt{\mathrm{Q}}\left\{\frac{s(2nv) - s(nv)}{is'(nv)} - \frac{s(4nv) + s(2nv) - 2s(nv)}{2is'(nv)}\right\}$$

$$= -\sqrt{\mathrm{Q}}\,\frac{s(4nv) - s(2nv)}{2is'(nv)} \qquad (22).$$

Now

$$\mathrm{M}\,dz = \frac{s(3nv) - s(nv)}{t^2}\,dt \qquad (23),$$

$$\sqrt{\mathrm{Z}} = \frac{\sqrt{\mathrm{Q}}\sqrt{\mathrm{S}}}{2t^2} \qquad (24),$$

so that

$$\frac{dz}{\sqrt{\mathrm{Z}}} = 2\,\frac{s(3nv) - s(nv)}{\mathrm{M}\sqrt{\mathrm{Q}}}\,\frac{ds}{\sqrt{\mathrm{S}}} \qquad (25),$$

and taking

$$\mathrm{M}\sqrt{\mathrm{Q}} = 2\,\{s(3nv) - s(nv)\} \qquad (26),$$

so that

$$\sqrt{\mathrm{Q}} = 2is'(nv) \qquad (27),$$

$$\mathrm{M} = \frac{s(3nv) - s(nv)}{is'(nv)} = \frac{is'(2nv)}{\{s(2nv) - s(nv)\}^2} \qquad (28),$$

we obtain from (17), (21), (22),

$$p = 4is'(2nv) \qquad (29),$$

$$a = \frac{s''(2nv)}{is'(2nv)} \qquad (30),$$

$$b = -s(4nv) + s(2nv) \qquad (31);$$

and ABEL's integral

$$\int \frac{z - k}{\sqrt{\mathrm{Z}}}\,dz = \int\left\{\frac{is'(nv)}{s(3nv) - s(nv)} - k - \frac{is'(nv)}{t}\right\}\frac{ds}{\sqrt{\mathrm{S}}}$$

$$= \int \frac{2\mathrm{P}(nv)\,t - is'(nv)}{s - s(nv)}\,\frac{ds}{\sqrt{\mathrm{S}}} = 2\mathrm{I}(nv) \qquad (32),$$

on taking

$$k = \frac{is'(nv)}{s(3nv) - s(nv)} - 2\mathrm{P}(nv)$$

$$= \mathrm{P}(2nv) - \mathrm{P}(4nv) = \tfrac{1}{2}a - \mathrm{P}(2nv) \qquad (33).$$

Putting $2nv = w$,

$$p = 4i\wp' w, \qquad a = \frac{\wp'' w}{i\wp' w}, \qquad ap = 4\wp'' w \qquad (34).$$

so that ABEL's recurring formula for q_m ('Œuvres,' 2, p. 157),

$$q_m + q_{m-2} = \frac{\frac{1}{2}p^2}{q_{m-1}{}^2} + \frac{ap}{q_{m-1}} \qquad (35),$$

becomes

$$\tfrac{1}{2}q_m + \tfrac{1}{2}q_{m-2} = -\frac{\wp'^2 w}{\frac{1}{4}q_{m-1}{}^2} + \frac{\wp'' w}{\frac{1}{2}q_{m-1}} \qquad (36),$$

and on comparison with the Weierstrassian formula,

$$\wp(m+1)w + \wp(m-1)w - 2\wp w = \frac{\wp'^2 w}{(\wp mw - \wp w)^2} + \frac{\wp'' w}{\wp mw - \wp w} \qquad (37),$$

we infer that

$$\tfrac{1}{2}q_{m-1} = -\wp mw + \wp w \qquad (38).$$

From another formula,

$$\wp(m+1)w - \wp w = \tfrac{1}{2}\frac{\wp'' w}{\wp mw - \wp w} - \tfrac{1}{2}\frac{\wp' w(\wp' mw - \wp' w)}{(\wp mw - \wp w)^2} \qquad (39),$$

ABEL's relation

$$\begin{aligned}\frac{c_{m-1}}{p_{m-1}} = \frac{q_m q_{m-1}}{p} &= \frac{\{\wp(m+1)w - \wp w\}(\wp mw - \wp w)}{i\wp' w} \\ &= \tfrac{1}{2}\frac{\wp'' w}{i\wp' w} + \tfrac{1}{2}i\frac{\wp' mw - \wp' w}{\wp mw - \wp w} \qquad (40),\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{2}i\frac{\wp' mw - \wp' w}{\wp mw - \wp w} &= i\zeta(m+1)w - i\zeta mw - i\zeta w \\ &= \mathrm{P}(m+1)w - \mathrm{P}mw - \mathrm{P}w \qquad (41);\end{aligned}$$

so that

$$g_{m-1} = a - \frac{c_{m-1}}{p_{m-1}} = \tfrac{1}{2}a - \mathrm{P}(m+1)w + \mathrm{P}(mw) + \mathrm{P}(w) \qquad (42).$$

Writing μ for ABEL's $n + 2$, his k is given by (2, p. 161)

$$\begin{aligned}\mu k &= (\mu - 1)a - (g_0 + g_1 + \ldots + g_{\mu-3}) \\ &= (\mu - 1)a - (\mu - 2)\tfrac{1}{2}a + \mathrm{P}(\mu - 1)w - (\mu - 1)\mathrm{P}w \\ &= \tfrac{1}{2}\mu a - \mu\mathrm{P}(w) \qquad (43),\end{aligned}$$

as before in (33), since

$$\mathrm{P}(\mu - 1)w = -\mathrm{P}w \qquad (44).$$

With

$$n = 1, \quad w = 2v,$$

$$\mathrm{M} = \frac{y}{x}, \quad p = 4xy, \quad a = \frac{2x - y - y^2}{y}, \quad b = -x\frac{y - x - y^2}{y^2} \qquad (45),$$

$$k = -2\mathrm{P}(v) + \frac{x}{y} \qquad (46).$$

But with

$$n = \tfrac{1}{2}(\mu - 1), \quad \wp w = \wp v,$$

$$\mathrm{M} = \frac{x}{\{s\,\tfrac{1}{2}(\mu-1)\,v + x\}^2}, \quad p = 4x, \quad a = 1 + y, \quad b = -x \qquad (47),$$

$$k = \mathrm{P}(v) - \mathrm{P}(2v) = \mathrm{P}(v) - \mathrm{P}(\mu - 1)\,v = 2\mathrm{P}(v) \qquad (48).$$

6. In the simplest case of

$$\mu = 3, \quad v = \tfrac{2}{3}\,\omega_3, \quad x = 0 \qquad (1),$$

the form (6), § 1, is illusory; but we take (L.M.S., 25, p. 210)

$$\mathrm{I}\,(\tfrac{2}{3}\,\omega_3) = \int \frac{\tfrac{1}{6}\,s + \tfrac{1}{2}\,c}{s\,\sqrt{\{4s^3 - (s+c)^2\}}}\,ds$$

$$= \frac{1}{3}\cos^{-1}\frac{s+c}{2s^{\frac{3}{2}}} = \frac{1}{3}\sin^{-1}\frac{\sqrt{\{4s^3 - (s+c)^2\}}}{2s^{\frac{3}{2}}} \qquad (2);$$

and putting $s = t^2$,

$$\mathrm{I} = \frac{2}{3}\cos^{-1}\frac{\sqrt{(2t^3 + t^2 + c)}}{2t^3} = \frac{2}{3}\sin^{-1}\frac{\sqrt{(2t^3 - t^2 - c)}}{2t^3} \qquad (3).$$

Writing ω for ω_3,

$$12\wp\,\tfrac{2}{3}\,\omega = -1, \quad i\wp'\,\tfrac{2}{3}\,\omega = c, \quad i\,(\zeta\,\tfrac{2}{3}\,\omega - \tfrac{2}{3}\,\eta) = \tfrac{1}{6}, \quad e^{-\frac{2\eta\omega}{9}}\,\sigma\,\tfrac{2}{3}\,\omega = c^{-\frac{1}{3}} \qquad (4).$$

In Abel's form ('Œuvres,' 2, p. 163),

$$\mathrm{J} = \int \frac{z + \tfrac{1}{3}\,a}{\sqrt{\{(z^2 + az)^2 + pz\}}}\,dz$$

$$= \tfrac{1}{3}\log\left[(z+a)^2 z + \tfrac{1}{2}\,p + (z+a)\sqrt{\{(z^2+az)^2 + pz\}}\right]$$

$$= \tfrac{2}{3}\log\left[(z+a)\sqrt{z} + \sqrt{\{(z+a)^2 z + p\}}\right] \qquad (5);$$

and with

$$z = y^2, \quad p = -q^2,$$

$$\mathrm{J} = \tfrac{4}{3}\log\{\sqrt{(y^3 + ay + q)} + \sqrt{(y^3 + ay - q)}\}$$

$$= \tfrac{4}{3}\,\mathrm{ch}^{-1}\sqrt{\frac{y^3 + ay + q}{2q}} = \tfrac{4}{3}\,\mathrm{sh}^{-1}\sqrt{\frac{y^3 + ay - q}{2q}} \qquad (6).$$

Or, in the circular form,

$$\mathrm{J} = \tfrac{4}{3}\cos^{-1}\sqrt{\frac{y^3 + ay + q}{2q}} = \tfrac{4}{3}\sin^{-1}\sqrt{\frac{-y^3 - ay + q}{2q}}$$

$$= -2\int \frac{y^2 + \tfrac{1}{3}\,a}{\sqrt{\{-y^2(y^2+a)^2 + q^2\}}}\,dy \qquad (7).$$

In the next case (L.M.S., 25, p. 213),

$$\mu = 5, \quad v = \tfrac{2}{5} r\omega, \quad s(5v) = \infty, \quad y = x \tag{1},$$

$$\begin{aligned} \mathrm{I}(v) &= \int \frac{\frac{1}{10}(3+x)(s+x) - \frac{1}{2}x}{(s+x)\sqrt{\mathrm{S}}}\,ds \\ &= \frac{1}{5}\cos^{-1}\frac{(3+x)(s+x)^2 - (1+2x)(s+x) + x}{2(s+x)^{\frac{5}{2}}} \\ &= \frac{1}{5}\sin^{-1}\frac{s+x-1}{2(s+x)^{\frac{5}{2}}}\sqrt{\mathrm{S}} \end{aligned} \tag{2},$$

$$\begin{aligned} \mathrm{I}(2v) &= \int \frac{\frac{1}{10}(1-3x)s - \frac{1}{2}x^2}{s\sqrt{\mathrm{S}}}\,ds \\ &= \frac{1}{5}\cos^{-1}\frac{(1-3x)s^2 - (2x^2 - x^3)s + x^4}{2s^{\frac{5}{2}}} \\ &= \frac{1}{5}\sin^{-1}\frac{s-x^2}{2s^{\frac{5}{2}}}\sqrt{\mathrm{S}} \end{aligned} \tag{3},$$

$$12\wp\tfrac{2}{5}\omega = -1 - 6x - x^2, \quad i\wp'\tfrac{2}{5}\omega = x, \quad \mathrm{P}\tfrac{2}{5}\omega = \frac{3+x}{10}, \quad e^{-\frac{2\eta\omega}{25}}\sigma\tfrac{2}{5}\omega = x^{-} \tag{4},$$

$$12\wp\tfrac{4}{5}\omega = -1 + 6x - x^2, \quad i\wp'\tfrac{4}{5}\omega = x^2, \quad \mathrm{P}\tfrac{4}{5}\omega = \frac{1-3x}{10}, \quad e^{-\frac{8\eta\omega}{25}}\sigma\tfrac{4}{5}\omega = x^{-} \tag{5},$$

and $x^{\frac{1}{5}}$ is the *icosahedron irrationality*, and KIEPERT'S $f^{-1} = x$.

In ABEL'S form, with $y = x$, Z has the factor $z - x$,

$$\mathrm{Z} = (z-x)[-z^3 + (2+x)z^2 - (1-2x)z + x] \tag{6},$$

and putting $z - x = \dfrac{x^2}{s}$, ABEL'S integral

$$\int \frac{z-k}{\sqrt{\mathrm{Z}}}\,dz = 2\mathrm{I}(2v), \text{ with } k = \frac{1+2x}{5} \tag{7}.$$

Putting $s + x = t^2$,

$$\begin{aligned} \mathrm{I}(v) &= \frac{2}{5}\cos^{-1}\frac{(t+1)\sqrt{\{2t^3 - (1-x)t^2 - 2xt + x\}}}{2t^{\frac{5}{2}}} \\ &= \frac{2}{5}\sin^{-1}\frac{(t-1)\sqrt{\{2t^3 + (1-x)t^2 - 2xt - x\}}}{2t^{\frac{5}{2}}} \end{aligned} \tag{8},$$

and the degree of the expressions is halved, with great gain of symmetry.

The degree is halved with greater ease by putting $s = t^2$ in $\mathrm{I}(2v)$, and now

$$\begin{aligned} \mathrm{I}(2v) &= \frac{2}{5}\cos^{-1}\frac{(t-x)\sqrt{\{2t^3 + (1+x)t^2 + 2xt + x^2\}}}{2t^{\frac{5}{2}}} \\ &= \frac{2}{5}\sin^{-1}\frac{(t+x)\sqrt{\{2t^3 - (1+x)t^2 + 2xt - x^2\}}}{2t^{\frac{5}{2}}} \end{aligned} \tag{9},$$

derivable from the preceding $\mathrm{I}(v)$ by writing $\dfrac{t}{x}$ for t, and $-\dfrac{1}{x}$ for x.

8. This suggests that in the general case of $\mu = 2n + 1$ it is simpler to work with

$$\mathrm{I}(2v) = \int \frac{\mathrm{P}(2v)\, s - \frac{1}{2} xy}{s \sqrt{\mathrm{S}}}\, ds \tag{1}$$

and to put $s = t^2$; and then with

$$\mathrm{T}_1 = 2t^3 + (1 + y)\, t^2 + 2xt + xy \tag{2}$$

$$\mathrm{T}_2 = 2t^3 - (1 + y)\, t^2 + 2xt - xy \tag{3}$$

we shall have

$$\mathrm{I}(2v) = \frac{2}{2n+1} \cos^{-1} \frac{t^{n-1} + h_1 t^{n-2} + \ldots + h_{n-1}}{2t^{n+\frac{1}{2}}} \quad (\sqrt{\mathrm{T}_1}, \text{ or } \sqrt{\mathrm{T}_2})$$

$$= \frac{2}{2n+1} \sin^{-1} \frac{t^{n-1} - h_1 t^{n-2} + \ldots + (-1)^{n-1} h_{n-1}}{2t^{n+\frac{1}{2}}} (\sqrt{\mathrm{T}_2}, \text{ or } \sqrt{\mathrm{T}_1}) \tag{4}$$

according as n is even or odd; and the results are of one-quarter the degree that would be given by ABEL's method of the periodic continued fraction; and since

$$(t^{n-1} + h_1 t^{n-2} + \ldots)^2 \mathrm{T}_1 + (t^{n-1} - h_1 t^{n-2} + \ldots)^2 \mathrm{T}_2 = 4t^{2n+1} \tag{5}$$

the determination of $h_1, h_2, \ldots$ can be carried out by a consideration of the *réduites* (HALPHEN, 'F.E.,' 2, p. 576) in preference to continued fractions, once the coefficients of t in T_1 and T_2 have been assigned.

9. Thus (L.M.S., 25, p. 222) for

$$\mu = 7, \quad v = \frac{2\omega}{7}, \quad \gamma_7 = 0, \quad \text{or} \quad xy - x^2 - y^3 = 0 \tag{1}$$

is a unicursal C_3, in which

$$x = z(1 - z)^2, \quad y = z(1 - z) \tag{2}$$

and

$$\mathrm{T}_1, \mathrm{T}_2 = 2t^3 \pm (1 + z - z^2)\, t^2 + 2z(1 - z)^2 t \pm z^2 (1 - z)^3 \tag{3}$$

$$\mathrm{P}(2v) = \frac{3 - 9z + 5z^2}{14} \tag{4}$$

$$\mathrm{I}(2v) = \int \frac{\mathrm{P}(2v)\, t^2 - \frac{1}{2} z^2 (1 - z)^3}{t^2} \frac{dt^2}{\sqrt{\mathrm{T}_1 \mathrm{T}_2}}$$

$$= \frac{2}{7} \cos^{-1} \frac{t^2 + h_1 t + h_2}{2t^{\frac{7}{2}}} \sqrt{\mathrm{T}_2}$$

$$= \frac{2}{7} \sin^{-1} \frac{t^2 - h_1 t + h_2}{2t^{\frac{7}{2}}} \sqrt{\mathrm{T}_1} \tag{5}$$

$$h_1 = (1 - z)^2, \quad h_2 = z(1 - z)^3 \tag{6}$$

Introducing a normalising homogeneity factor M, so that the substitution $\left(z, \frac{1}{1-z}, \frac{z-1}{z}\right)$ should correspond to $(r, 2r, 4r)$,

$$\mathrm{M} = z^{\frac{2}{3}}(1-z)^{\frac{2}{3}} \tag*{(7),}$$

derived from (L.M.S., 27, p. 453),

$$\mathrm{M}^3 = \mathrm{M}_1\mathrm{M}_2\mathrm{M}_4, \quad \mathrm{M}_p = \frac{is'(pr)}{s(2pr) - s(pr)} = x \frac{\gamma_{2p}{}^3}{\gamma_p{}^3\gamma_{3p}} \tag*{(8),}$$

$$\frac{12\wp v}{\mathrm{M}^2} = \frac{-1 - 6z + 9z^2 - 2z^3 - z^4}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}}, \quad \frac{i\wp' r}{\mathrm{M}^3} = \frac{1}{z}, \quad \frac{\mathrm{P}(r)}{\mathrm{M}} = \frac{5 - z - z^2}{14z^{\frac{2}{3}}(1-z)^{\frac{2}{3}}} \tag*{(9),}$$

$$\frac{12\wp 2r}{\mathrm{M}^2} = \frac{-1 + 6z - 15z^2 + 10z^3 - z^4}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}}, \quad \frac{i\wp' 2r}{\mathrm{M}^3} = 1 - z, \quad \frac{\mathrm{P}(2r)}{\mathrm{M}} = \frac{3 - 9z + 5z^2}{14z^{\frac{2}{3}}(1-z)^{\frac{2}{3}}} \tag*{(10),}$$

$$\frac{12\wp 4r}{\mathrm{M}^2} = \frac{-1 + 6z - 3z^2 - 2z^3 - z^4}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}}, \quad \frac{i\wp' 4r}{\mathrm{M}^3} = \frac{z}{z-1}, \quad \frac{\mathrm{P}(4r)}{\mathrm{M}} = \frac{-1 + 3z + 3z^2}{14z^{\frac{2}{3}}(1-z)^{\frac{2}{3}}} \tag*{(11),}$$

$$\exp\frac{-2\eta\omega}{49}\,\sigma\frac{2\omega}{7} = z^{-\frac{1}{7}}(1-z)^{-\frac{4}{7}}, \quad \exp\frac{-8\eta\omega}{49}\,\sigma\frac{4\omega}{7} = z^{-\frac{1}{7}}(1-z)^{-\frac{2}{7}},$$

$$\exp\frac{-18\eta\omega}{49}\,\sigma\frac{6\omega}{7} = z^{-\frac{8}{7}}(1-z)^{-\frac{1}{7}} \tag*{(12).}$$

In the notation of KLEIN-FRICKE, 'Modulfunctionen,' 2, p. 399,

$$7\mathrm{B}_0{}^2 = -\frac{(1 - z + z^2)^2}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}} = -4\left\{\frac{\mathrm{P}(r) + \mathrm{P}(2r) + \mathrm{P}(4r)}{\mathrm{M}}\right\}^2$$

$$= 4\frac{\wp r + \wp 2r + \wp 4r}{\mathrm{M}^2} = 4\frac{\mathrm{G}_1}{\mathrm{M}^2} \tag*{(13),}$$

$$\tau = \frac{1 - 8z + 5z + z^2}{z(1-2)} = 5 + z + \frac{1}{1-z} + \frac{z-1}{z} \tag*{(14).}$$

10. With (L.M.S., 25, p. 232)

$$\mu = 9, \quad v = \frac{2r\omega}{9} \tag*{(1),}$$

$\gamma_9 = 0$ is a unicursal C_5, in which

$$x = p^2(1-p)(1-p+p^2), \quad y = p^2(1-p) \tag*{(2),}$$

$$\mathrm{T}_1, \mathrm{T}_2 = 2t^3 \pm (1 + p^2 - p^3)t^2 + 2p^2(1-p)(1-p+p^2)t \pm p^4(1-p)^2(1-p+p) \tag*{(3),}$$

and from the relation

$$\mathrm{P}(4v) + \mathrm{P}(5v) = 0 \tag*{(4),}$$

$$9\mathrm{P}(v) = 1 + \tfrac{1}{2}(1+y) + \frac{x}{y} + \frac{y^2}{y-x} \tag*{(5),}$$

$$18\mathrm{P}(2v) = 4 + 2(1+y) + 4\frac{x}{y} + \frac{4y^2}{y-x} - 9(1+y)$$

$$= 1 + 0 - 3p^2 + 7p^3 \tag*{(6),}$$

$$
\begin{aligned}
\mathrm{I}\,(2v) &= \int \frac{\mathrm{P}\,(2v)\,t^2 - \tfrac{1}{2}xy}{t^2}\,\frac{dt^2}{\sqrt{\mathrm{T}_1\mathrm{T}_2}} \\
&= \frac{2}{9}\cos^{-1}\frac{t^3 + h_1t^2 + h_2t + h_3}{2t^3}\,\sqrt{\mathrm{T}_1} \\
&= \frac{2}{9}\sin^{-1}\frac{t^3 - h_1t^2 + h_2t - h_3}{2t^3}\,\sqrt{\mathrm{T}_2}. \qquad (7)
\end{aligned}
$$

$$h_1 = -p^2(1-2p), \quad h_2 = -p^4(1-p+p^2), \quad h_3 = p^6(1-p)(1-p+p^2) \qquad (8).$$

The substitution $\left(p, \frac{p-1}{p}, \frac{1}{1-p}\right)$ will correspond to $(v, 2v, 4v)$ with a normalising factor

$$\mathrm{M} = (\mathrm{M}_1\mathrm{M}_2\mathrm{M}_4)^{\frac{1}{3}} = p\,(1-p) \qquad (9),$$

and

$$\frac{12\wp v}{\mathrm{M}^2} = \frac{-1+0-6p^2+10p^3-9p^4+6p^5-p^6}{p^2(1-p)^2}, \quad \frac{i\wp' v}{\mathrm{M}^3} = \frac{1-p+p^2}{p\,(1-p)^2} \qquad (10),$$

$$\frac{12\wp 2v}{\mathrm{M}^2} = \frac{-1+0+6p^2-14p^3+15p^4-6p^5-p^6}{p^2(1-p)^2}, \quad \frac{i\wp' 2v}{\mathrm{M}^3} = \frac{p\,(1-p+p^2)}{1-p} \qquad (11),$$

$$\frac{12\wp 4v}{\mathrm{M}^2} = \frac{-1+12p-30p^2+34p^3-21p^4+6p^5-p^6}{p^2(1-p)^2}, \quad \frac{i\wp' 4v}{\mathrm{M}^3} = \frac{(1-p)(1-p+p^2)}{p^2} \qquad (12);$$

while

$$\frac{12\wp 3v}{\mathrm{M}^2} = \frac{-(-1+0+3p^2-p^3)^2}{p^2(1-p)^2} = -\left[\frac{6\mathrm{P}\,(3v)}{\mathrm{M}}\right]^2, \quad \frac{i\wp' 3v}{\mathrm{M}^3} = 1 \qquad (13),$$

unchanged by the substitution.

Also

$$\mathrm{G}_1 = \frac{\wp v + \wp 2v + \wp 3v + \wp 4v}{\mathrm{M}^2} = \frac{(-1+p-p^2)^3}{3p^2(1-p)^2} \qquad (14),$$

a quantity required in the transformation of the ninth order; and

$$\frac{\mathrm{P}\,(v)}{\mathrm{M}} = \frac{5+0+3p^2-p^3}{18p\,(1-p)} \qquad (15),$$

$$\frac{\mathrm{P}\,(2v)}{\mathrm{M}} = \frac{1+0-3p^2+7p^3}{18p\,(1-p)} \qquad (16),$$

$$\frac{\mathrm{P}\,(4v)}{\mathrm{M}} = \frac{-7+18p-15p^2+5p^3}{18p\,(1-p)}$$

$$\frac{\mathrm{P}\,(3v)}{\mathrm{M}} = \frac{-1+0+3p^2-p^3}{6p\,(1-p)} = \frac{\eta}{6} = \frac{\xi+3}{6} \qquad (17),$$

(Kiepert, 'Math. Ann.,' 32, p. 66).

11. $$\mu = 11, \quad v = \frac{2r\omega}{11}, \quad \gamma_{11} = 0$$

(L.M.S., 25, p. 241; 'Math. Ann.,' 52, p. 484), leads by the substitutions

$$x = y(1 - z), \quad z - y = \frac{z^2}{1 + c} \tag{1}$$

to the bicursal C_3

$$z(1 - z) + c(c + 1)^2 = 0 \tag{2},$$

$$1 - 2z = \sqrt{C}, \quad C = 4c(c + 1)^2 + 1 \tag{3};$$

$$x = -\tfrac{1}{2}c(1 + c)(1 + 2c + \sqrt{C}) \tag{4},$$

$$y = \frac{-c - 4c^2 - 2c^3 - c\sqrt{C}}{2(1 + c)}, \quad 1 + y = \frac{2 + c - 4c^2 - 2c^3 - c\sqrt{C}}{2(1 + c)} \tag{5}.$$

We now find

$$\mathrm{P}(2v) = \frac{6 + 27c + 44c^2 + 18c^3 + (8 + 13c)\sqrt{C}}{44(1 + c)} \tag{6},$$

$$\begin{aligned}\mathrm{I}(2v) = \int \frac{\mathrm{P}(2v)\,t^2 - \frac{1}{2}xy}{t^2}\frac{dt^2}{\sqrt{\mathrm{T}_1\mathrm{T}_2}} &= \frac{2}{11}\cos^{-1}\frac{t^4 + h_1t^3 + h_2t^2 + h_3t + h_4}{2t^2}\sqrt{\mathrm{T}_2} \\ &= \frac{2}{11}\sin^{-1}\frac{t^4 - h_1t^3 + h_2t^2 - h_3t + h_4}{2t^2}\sqrt{\mathrm{T}_1}\end{aligned} \tag{7},$$

$$h_1 = \frac{2 + 7c + 10c^2 + 4c^3 + (2 + 3c)\sqrt{C}}{2(1 + c)} \tag{8},$$

$$h_2 = -\frac{1 + 10c + 40c^2 + 82c^3 + 86c^4 + 40c^5 + 6c^6 + (1 + 2c)^2(1 + 4c + 2c^2)\sqrt{C}}{2(1 + c)^2} \tag{9},$$

$$h_3 = -\tfrac{1}{2}[1 + 8c + 28c^2 + 52c^3 + 50c^4 + 20c^5 + 2c^6 + (1 + 2c)(1 + 4c + 6c^2 + 2c^3)\sqrt{C}] \tag{10},$$

$$\begin{aligned}h_4 = \frac{c}{2(1+c)}[1 + 11 + 54 + 151 + 255 + 254 + 135 + 32 + 2 + (1 + c)^2 \\ (1 + 7 + 19 + 22) + 5)\sqrt{C}]\end{aligned} \tag{11},$$

using detached coefficients.

We find also

$$\mathrm{P}(rv) = \frac{\mathrm{P}_r + \mathrm{Q}_r\sqrt{C}}{44(1 + c)}, \quad 12\wp rv = \frac{a_r + b_r\sqrt{C}}{2(1 + c)^2} \tag{12},$$

where

$$\begin{array}{ll} \mathrm{P}_1 = 14 + 19c + 0 - 2c^3, & \mathrm{Q}_1 = 4 + c, \\ \mathrm{P}_2 = 6 + 27c + 44c^2 + 18c^3, & \mathrm{Q}_2 = 8 + 13c, \\ \mathrm{P}_3 = -2 + 13c + 0 - 6c^3, & \mathrm{Q}_3 = 12 + 3c, \\ \mathrm{P}_4 = 12 + 43c + 44c^2 + 14c^3, & \mathrm{Q}_4 = -6 - 9c, \\ \mathrm{P}_5 = 4 + 7c + 0 - 10c^3, & \mathrm{Q}_5 = -2 + 5c \end{array} \tag{13}$$

and the values of a_r and b_r are given in L.M.S., 27, p. 455.

The normalising factor is

$$M = (M_1 M_2 M_3 M_4 M_5)^{\frac{1}{5}} \tag{14},$$

but so far this has not made evident any symmetry of results; it will be noticed that our parameter c here is an elliptic function of (L.M.S., 27, p. 429)

$$u = \int_c^\infty \frac{dc}{\sqrt{C}} \tag{15},$$

which is $\frac{1}{25}$th of a period out of phase with that required to lead to KLEIN's results.

12. $\mu = 13$, $v = \frac{2r\omega}{13}$, $\gamma_{13} = 0$ (L.M.S., 25, p. 251; 'Math. Ann.,' 52, p. 484) by the substitutions

$$x = y(1 - z), \quad z - y = \frac{z^2}{p}, \quad z = c(p - 1) \tag{1},$$

leads to a C_1 with class $p = 2$, in which

$$2p = 1 - c^2 - c^3 + \sqrt{C} \tag{2},$$

$$\begin{aligned} C &= 1 + 4c + 6c^2 + 2c^3 + c^4 + 2c^5 + c^6 \\ &= (1 + 2c - c^2 - c^3)^2 + 4c^2(1 + c)^2 \end{aligned} \tag{3},$$

and we find, using detached coefficients of ascending powers of c,

$$P(2r) = \frac{6 + 12 - 9 - 33 + 4 + 8 - 18 - 11 + (4 + 0 - 15 + 7 + 11)\sqrt{C}}{52(1 + c)^2} \tag{4},$$

$$\begin{aligned} I(2r) &= \int \frac{P(2r)t^2 - \frac{1}{2}xy}{t^2} \frac{dt^2}{\sqrt{T_1 T_2}} \\ &= \frac{2}{13}\cos^{-1}\frac{t^5 + h_1 t^7 + \ldots + h_5}{2t^{\frac{13}{2}}}\sqrt{T_1} \\ &= \frac{2}{13}\sin^{-1}\frac{t^5 - h_1 t^7 + \ldots - h_5}{2t^{\frac{13}{2}}}\sqrt{T_2} \end{aligned} \tag{5},$$

$$h_1 = \frac{1 + 2 - 3 - 9 + 1 + 2 - 5 - 3c^7 + (1 + 0 - 4 + 2 + 3c^4)\sqrt{C}}{2(1 + c)^2} \tag{6},$$

$$h_2 = \frac{A_2 + B_2\sqrt{C}}{2(1 + c)^7},$$

$$A_2 = -1 - 4 - 1 + 16 + 15 - 26 - 28 + 28 + 14 - 38 - 20 + 6 - 5 - 10 - 3c^{14},$$

$$B_2 = -1 - 2 + 4 + 9 - 7 - 15 + 12 + 14 - 7 - 2 + 7 + c'' \tag{7},$$

$$h_3 = \frac{c^2}{2(1 + c)^6}(A_3 + B_3\sqrt{C}),$$

$$\begin{aligned} A_3 = 1 + 6 + 7 - 26 - 64 + 24 + 154 - 6 - 222 + 32 + 266 \\ + 10 - 109 + 104 + 143 + 22 + 4 + 32 + 21 + 4c^{19}, \end{aligned}$$

$$\begin{aligned} B_3 = 1 + 4 - 2 - 25 - 10 + 61 + 27 - 97 - 28 \\ + 90 - 7 - 82 - 12 + 15 - 15 - 17 - 4c^{16} \end{aligned} \tag{8},$$

$$h_4 = \frac{c^4}{2(1+c)^5}(A_4 + B_4\sqrt{C}),$$

$$\begin{aligned} A_4 = 1 &+ 5 + 3 - 21 - 21 + 56 + 46 - 114 - 39 + 158 - 3 - 135 \\ &+ 52 + 91 - 37 - 22 + 35 + 14 - 5 + 2 + 4 + c^{21}, \end{aligned}$$

$$\begin{aligned} B_4 = 1 &+ 3 - 4 - 15 + 14 + 35 - 41 - 37 + 65 \\ &+ 12 - 61 + 8 + 31 - 16 - 14 + 5 + 1 - 3 - c^{18} \qquad (9), \end{aligned}$$

$$h_5 = -\frac{c^6}{2(1+c)^7}(A_5 + B_5\sqrt{C}),$$

$$\begin{aligned} A_5 = 1 &+ 7 + 12 - 22 - 74 + 38 + 223 - 76 - 448 + 205 + 614 - 403 - 551 \\ &+ 555 + 365 - 442 - 68 + 353 + 47 - 99 + 57 + 76 + 7 - 1 + 10 + 6 + c^{26}, \end{aligned}$$

$$\begin{aligned} B_5 = 1 &+ 5 + 1 - 28 - 16 + 91 + 35 - 205 - 2 + 301 - 97 - 290 \\ &+ 169 + 120 - 176 - 79 + 90 + 0 - 54 - 7 + 7 - 5 - 5 - c^{23} \qquad (10). \end{aligned}$$

By means of an appropriate homogeneity factor M, we can express

$$\frac{52\,P(rv)}{M} = P_r + Q_r\sqrt{C}, \quad \frac{12\wp rv}{M^2} = a_r + b_r\sqrt{C} \qquad (11)$$

in such a manner that the substitutions

$$\left(c,\ -\frac{1+c}{c},\ -\frac{1}{1+c}\right) \text{ correspond to } (v,\ 3v,\ 9v)$$

and

$$(\sqrt{C},\ -\sqrt{C}) \text{ to } (v,\ 5v)$$

(L.M.S., 25, p. 255; 27, p. 416), and

$$M = (M_1M_2M_3M_4M_5M_6)^{\frac{1}{6}} \text{ or } (M_1M_5)^{\frac{1}{2}} \qquad (12).$$

13. $\mu = 15$, $v = \frac{2r\omega_3}{15}$, (L.M.S., 25, p. 258; 'Math. Ann.,' 52, p. 485), $\gamma_{15} = 0$ is reducible to a bicursal C_4.

Changing c in L.M.S., 25, p. 259, into $b-1$, and normalising by a homogeneity factor

$$M = (M_1M_4)^{\frac{1}{2}} = \left\{\frac{\sqrt{(b^2+b+1)} + \sqrt{(b^2-3b+1)}}{2}\right\}^3 \qquad (1),$$

$$\begin{aligned} I(2v) &= \int \frac{Pt^2 - \frac{1}{2}S}{t^2}\frac{dt^2}{\sqrt{T_1T_2}} \\ &= \frac{2}{15}\cos^{-1}\frac{t^6 - H_1t^5 + \ldots + H_6}{2t^{\frac{15}{2}}}\sqrt{T_2} \\ &= \frac{2}{15}\sin^{-1}\frac{t^6 + H_1t^5 + \ldots + H_6}{2t^{\frac{15}{2}}}\sqrt{T_1} \qquad (2), \end{aligned}$$

$$T_1, T_2 = 2t^3 \pm Qt^2 + 2Rt \pm S \qquad (3)$$

$$Q = \frac{1+y}{M} =$$

$$\frac{-(b-1)(b^6-2b^5-b^4-b^3+b^2+0+1)-(b+1)(b^4-3b^3+3b^2-b+1)\sqrt{B}}{2b^3\sqrt{(b^2+b+1)}} \quad (4),$$

$$B = (b^2+b+1)(b^2-3b+1) \quad (5),$$

$$R = \frac{x}{M^2} = \frac{(b-1)\{(b^8-4+3+3-2-3+2+2-1)\sqrt{(b^2+b+1)} + (b^8-2-1+1+2-1-2+0+1)\sqrt{(b^2-3b+1)}\}}{2b^4\sqrt{(b^2+b+1)}} \quad (6),$$

$$S = \frac{xy}{M^3} = -\frac{(b-1)^2}{2b^5(b^2+b+1)}$$

$$\{(b^{12}-6+10+1-10-6+12+6-9-5+4+2-1)\sqrt{(b^2+b+1)}$$
$$+(b^{12}-4+2+5+2-8-4+6+5-3-4+0+1)\sqrt{(b^2-3b+1)}\} \quad (7),$$

$$P = \frac{P(2x)}{M} = \frac{(b-1)(13b^6-30-9+3+21+0-7)+(13b^5-30+4+14+0-7)\sqrt{B}}{60b^3\sqrt{(b^2+b+1)}} \quad (8).$$

Differentiating (2) and equating coefficients of t we find

$$H_1 = -\tfrac{1}{4}(15P+Q)$$
$$= \frac{(b-1)(-3b^6+7+2-1-5+0+2)+(-3b^5+7-1-3+0+2)\sqrt{B}}{2b^3\sqrt{(b^2+b+1)}} \quad (9);$$

$$H_2 = -\tfrac{1}{2}R + \tfrac{1}{8}(15P-3Q)H$$
$$= \frac{(b-1)(M_2+N_2\sqrt{B})}{2b^6(b^2+b+1)},$$

$$M_2 = -6b^{13}+33-49+0+8+58-22-39+0+24+4-8-1+1,$$
$$N_2 = -6b^{11}+27-28-13+9+28-7-18+1+7+0-1 \quad (10)$$

$$H_3 = \frac{(b-1)^2(M_3+N_3\sqrt{B})}{2b^9(b^2+b+1)^2},$$

$$M_3 = 4b^{19}-31+80-61-21-54+159+16-113-85$$
$$+97+90-41-64+7+29+2-8-1+1,$$
$$N_3 = +4b^{17}-27+57-23-33-34+84+30-65-46$$
$$+42+40+15-21+3+7+0-1 \quad (11);$$

$$H_4 = \frac{(b-1)^3(M_4+N_4\sqrt{B})}{2b^{18}(b^2+b+1)^2},$$

$$M_4 = 5b^{23}-49+175-247+48+8+468-380-435+93+684$$
$$+48-576-234+350+252-124-150+18+54+3-11-1+1,$$
$$N_4 = 5b^{21}-44+136-145-39+28+303-145-325-19$$
$$+362+69-267-120+131+93-38-41+5+10+0-1 \quad (12);$$

$$\mathrm{H}_5 = \frac{(b-1)^4 \{\mathrm{M}_5 \sqrt{(b^2+b+1)} + \mathrm{N}_5 \sqrt{(6^2-3b+1)}\}}{2b^{11}(b^2+b+1)^2}.$$

$$\mathrm{M}_5 = -b^{25}+13-68+176-209+43+20+305-359-235+298+413$$
$$-308-437+191+386-70-260-3+130+21-44-11+9+2b-1,$$

$$\mathrm{N}_5 = -b^{25}+11-46+84-43-23-100+215+59-197-210+231$$
$$+274-143-297+34+230+40-127-54+45+32-7-9b^2+0+1 \quad (13):$$

$$\mathrm{H}_6 = -\frac{\mathrm{S}}{\mathrm{R}}\mathrm{H}_5 = \frac{(b-1)^6 \{\mathrm{M}_6 \sqrt{(b^2+b+1)} + \mathrm{N}_6 \sqrt{(b^2-3b+1)}\}}{2b^{12}(b^2+b+1)^2},$$

$$\mathrm{M}_6 = -b^{29}+15-93+299-494+298+76+387-1151+128+1216$$
$$+370-1762-619+1720+982-1349-1086+773+891$$
$$-288-540+36+232+25-65-13+11+2b-1,$$

$$\mathrm{N}_6 = -b^{29}+13-67+165-166-10-56+513-285-588+26+1062$$
$$+152-1165-560+1014-827-594-815+169+562+64$$
$$-266-100+75+49-9-11b^2+0+1 \quad (14).$$

These calculations, as well as for $\mu = 11$ and 13, and their verification, were carried out for me by Mr. J. W. Hicks, of Greenwich Observatory.

Putting

$$\frac{12\wp rv}{\mathrm{M}^2} = \frac{a_r + b_r\sqrt{\mathrm{B}}}{2b^6(b^2+b+1)} \quad (15),$$

then, since

$$\frac{12\wp 2v}{\mathrm{M}^2} = -\mathrm{Q}^2 + 8\mathrm{R} \quad (16),$$

we find

$$a_2 = -b^{14}+14-43+28+19+22-54-12+30+22-17-8+5+2-1 \quad (17),$$

$$b_2 = -(b-1)(b^{11}-12+19+11-9-19+5+15-1-5+0+1) \quad (18).$$

The substitution $\left(b, \frac{1}{b}\right)$ changes v into $4v$, $2v$ into $8v$, . . ., so that a_8 and b_8 are obtained from a_2 and b_2 by writing the coefficients in reverse order.

$$a_8 = -b^{14}+2+5-8-17+22+30-12-54+22+19+28-43+14-1 \quad (19),$$

$$b_8 = -(b-1)(b^{11}+0-5-1+15+5-19-9+11+19-12+1) \quad (20).$$

Again, since

$$\frac{12\wp v}{\mathrm{M}^2} = -\mathrm{Q}^2 - 4\mathrm{R} \quad (21),$$

we find

$$a_1 = -b^{14}+2+5-8-5-2+18+0-18-2+7+16-7+2-1 \quad (22)$$

$$b_1 = -(b-1)(b^{11}+0-5-1+3+5-7-9-1+7+0+1) \quad (23).$$

and in a_7, b_7 the coefficients run in reverse order.

So also

$$\frac{\mathrm{P}(r)}{\mathrm{M}} = \frac{(b-1)(-b^6+0+3+9+3+0-11) + (-b^5+0+2-8-0-11)\sqrt{\mathrm{B}}}{60b^3\sqrt{(b^2+b+1)}} \qquad (24),$$

and P (4*r*), P (8*r*) are obtained from P (*r*), P (2*r*) by writing the coefficients in reverse order.

We find also

$$a_3 = -b^{14} + 2 + 5 - 8 - 5 - 2 + 6 + 12 + 6 - 2 - 5 - 8 + 5 + 2 - 1,$$
$$b_3 = -(b^2-1)(b^4 - b^3 - b^2 - b + 1)(b^6 + 0 - 3b^4 + b^3 - 3b^2 + 0 + 1) \qquad (25);$$

$$a_6 = -b^4 + 2 + 5 + 4 - 29 - 26 + 18 + 60 + 18 - 26 - 29 + 4 + 5 + 2 - 1,$$
$$b_6 = -(b^2-1)(b^4 - b^3 - b^2 - b + 1)(b^6 + 0 - 3b^4 - 11b^3 - 3b^2 + 0 + 1) \qquad (26);$$

$$\frac{\mathrm{P}(3r)}{\mathrm{M}} = \frac{-(b-1)(b^6+0-3+1-3+0+1) - (b+1)(b^4-b^3-b^2-b+1)\sqrt{\mathrm{B}}}{20b^3\sqrt{(b^2+b+1)}} \qquad (27),$$

$$\frac{\mathrm{P}(6r)}{\mathrm{M}} = \frac{(b-1)(3b^6+0-9-17-9+0+3) + 3(b+1)(b^4-b^3-b^2-b+1)\sqrt{\mathrm{B}}}{20b^3\sqrt{(b^2+b+1)}} \qquad (28),$$

and these are unchanged by the substitution $\left(b, \frac{1}{b}\right)$.

Also

$$\frac{\mathrm{P}(5r)}{\mathrm{M}} = \frac{-(b^3+1)(b^4-b^3-3b^2-b+1) - (b-1)(b^4+b^3-b^2+b+1)\sqrt{\mathrm{B}}}{12b^3\sqrt{(b^2+b+1)}} \qquad (29)$$

and

$$\frac{12\wp 5r}{\mathrm{M}^2} = -\left[\frac{6\mathrm{P}(5r)}{\mathrm{M}}\right]^2 \qquad (30).$$

With

$$\mathrm{M}' = (\mathrm{M}_1\mathrm{M}_2\mathrm{M}_4\mathrm{M}_8)^{\frac{1}{4}} \qquad (31),$$

$$\mathrm{M}'^2 = \frac{b^3(b-1)}{\sqrt{(b^2+b+1)}}\left[\frac{(b-2)\sqrt{(b^2+b+1)} + b\sqrt{(b^2-3b+1)}}{2}\right]^3 \qquad (32),$$

the expressions for $\frac{12\wp rv}{\mathrm{M}'^2}$ are lowered in degree ; for

$$\frac{\mathrm{M}^2}{\mathrm{M}'^2} = \frac{\sqrt{(b^2+b+1)}}{2b^3(b-1)}\left[(b^5-2-2+2+2-1)\sqrt{(b^2+b+1)}\right.$$
$$\left. - (b^5+0-2-2+0+1)\sqrt{(b^2-3b+1)}\right] \qquad (33),$$

$$\frac{12\wp rv}{\mathrm{M}'^2} = \frac{(b-1)p_r\sqrt{(b^2+b+1)} + bq_r\sqrt{(b^2-3b+1)}}{2b^3(b-1)\sqrt{(b^2+b+1)}} \qquad (34),$$

$$p_1 = -2b^6 + \ 3 + 3 + 1 - 9 - 9 - 2, \qquad q_1 = 3(-b^5 + 0 - 2 + 2 + 0 + 3) \quad (35),$$

$$p_4 = -2 \ \ - \ 9 - 9 + 1 + 3 + 3 - 2, \qquad q_4 = 3(3 + 0 + 2 - 2 + 0 - 1) \quad (36),$$

$$p_2 = -2 \ \ + 15 - 21 + 13 - 9 + 3 - 2, \quad q_2 = 3(3 - 4 + 2 - 2 + 4 - 1) \quad (37),$$

$$p_8 = -2 \ \ + \ 3 - 9 + 13 - 21 + 15 - 2, \quad q_8 = 3(-1 + 4 - 2 + 2 - 4 + 3) \quad (38),$$

$$p_1 + p_4 + p_2 + p_8 = -4(2b^2 - b + 2)(b^4 - b^3 + 3b^2 - b + 1) \quad (39).$$

$$q_1 + q_4 + q_2 + q_8 = 12(b^5 + 1) \quad (40);$$

$$p_3 = -2 + 3 + 3 - 11 + 3 + 3 - 2, \quad q_3 = 3(b + 1)(-b^4 + 1 + 1 + 1 - 1) \quad (41),$$

$$p_6 = -2 + 3 + 3 + 13 + 3 + 3 - 2, \quad q_6 = 3(b + 1)(-b^4 + 1 + 1 + 1 - 1) \quad (42),$$

so that

$$\wp 6v - \wp 3v = \mathrm{M}'^2 \quad (43);$$

$$p_5 = -2 + 3 + 3 + 1 + 3 + 3 - 2, \quad q_5 = 3(b + 1)(-b^4 + 1 - 3 + 1 - 1) \quad (44),$$

14. The case of $\mu = 15$ can also be derived by a trisection of $\mu = 5$, and so generally when $\mu = 3n$ is any multiple of 3.

For when

$$\mathrm{S} = 4s(s + x)^2 - \{(1 + y)s + xy\}^2 \quad (1)$$

is expressed in the form

$$\mathrm{S} = 4(s + t)^3 - (\mathrm{A}s + \mathrm{B})^2 \quad (2),$$

this implies that

$$s(\tfrac{2}{3}\omega) = -t, \quad is'(\tfrac{2}{3}\omega) = \mathrm{A}t - \mathrm{B} \quad (3);$$

and now

$$\mathrm{A}^2 = 12t + (1 + y)^2 - 8x = -12\wp\tfrac{2}{3}\omega \quad (4),$$

$$2\mathrm{AB} = 12t^2 - 4x^2 + 2xy(1 + y) \quad (5),$$

$$\mathrm{B}^2 = 4t^3 + x^2y^2 \quad (6),$$

so that

$$\{12t + (1 + y)^2 - 8x\}(4t^3 + x^2y^2) - (6t^2 - 2x^2 + xy + xy^2)^2 = 0 \quad (7),$$

reducing to

$$3t^4 + \{(1 + y)^2 - 8x\}t^3 + 3x(2x - y - y^2)t^2 + 3x^2y^2t + x^3(y - x - y^2) = 0 \quad (8),$$

a Jacobian quartic for t, which can be resolved.

For

$$n = 5, \quad y = x,$$

$$3t^4 + (x^2 - 6x + 1)t^3 - 3x^2(x - 1)t^2 + 3x^4t - x^5 = 0 \quad (9),$$

and putting $t = cx$,

$$(c - 1)^3x^2 + 3c^2(c - 1)^2x + c^3 = 0 \quad (10).$$

To identify with the preceding results in § 13, put

$$c = \frac{b^2+b+1}{3b},\quad c-1 = \frac{(b-1)^2}{3b},\quad 3c+1 = \frac{(b+1)^2}{b},\quad 3c-4 = \frac{b^2-3b+1}{b} \qquad (11),$$

and then the associated octahedron irrationality $o = \frac{1}{2}$.

15. For $\mu = 17$, the quartic for q in terms of c is given in L.M.S., 25, p. 264, obtained by the substitution in $\gamma_{17} = 0$ of

$$x = y(1-z),\quad y = z\left(1-\frac{z}{p}\right),\quad z = \frac{c(1+c)}{q-1},\quad p = \frac{q+c}{q-1} \qquad (1).$$

This irreducible quartic is made reducible by putting $q = -c(e+1)$, and with $c = b-1$ becomes

$$e(e+1)b^4 + e(2e^2+2e+1)b^3 + (e^4-e^3-3e^2-3e-1)b^2$$
$$- e(e+1)(2e^2+2e+1)b + e(e+1)^3 = 0 \qquad (2),$$

$$\left\{b - \frac{e+1}{b} + \frac{2e^2+2e+1}{2(e+1)}\right\}^2 = \frac{4e^2+9e+4}{4e(e+1)^2} \qquad (3).$$

The alternating function

$$\frac{s(8r)-s(2r)}{s(4r)-s(\ r)} = \frac{(b-1)(b+c+1)}{eb} = \frac{-\sqrt{e}+\sqrt{(4e^2+9e+4)}}{2(e+1)e^{\frac{1}{2}}} \qquad (4),$$

and the division values are associated with elliptic functions of an argument

$$u = \int_e^\infty \frac{de}{\sqrt{(4e^3+9e^2+4e)}} \qquad (5),$$

$$e(u+\omega) = \frac{1}{e},\quad e(\tfrac{1}{2}\omega) = 1,\quad e(\omega'+\tfrac{1}{2}\omega) = -1,\quad o = \frac{\sqrt{17}-1}{4} \qquad (6),$$

$$f = e(u+\omega'+\tfrac{1}{2}\omega) = -\left\{\frac{-\sqrt{e}+\sqrt{(4e^2+9e+4)}}{2(e+1)}\right\}^2,\quad (e+1)^2(f+1)^2+ef = 0 \quad (7).$$

By the transformation

$$4e+9+\frac{4}{e} = t^2,\quad \frac{4(e+1)^2}{e} = t^2-1,\quad \frac{4(e-1)^2}{e} = t^2-17 \qquad (8),$$

$$u = \int \frac{dt}{\sqrt{(t^2-1\,.\,t^2-17)}} \qquad (9),$$

and a comparison can be made with the equations of KIEPERT ('Math. Ann.,' 37, p. 386)

16. The next case of $\mu = 19$ presents difficulties not yet surmounted, although it was hoped that the analogy with $\mu = 11$ would give the clue required.

In the case of $\mu = 11$, the substitution of

$$1 - z = (1+c)\left(1+\frac{1}{a}\right) \qquad (1)$$

in (2), § 11, makes the ellipsotomic relation $\gamma_{11} = 0$ equivalent to

$$a^2c^2 - 2ac - a - c - 1 = 0 \qquad (2),$$

an addition equation of the elliptic functions a and c of the argument

$$u = \int_c^\infty \frac{dc}{\sqrt{\mathrm{C}}}, \quad \mathrm{C} = 1 + 4c(1 + c)^2 \qquad (3),$$

a and c differing in phase by one-fifth of a period; and the five division values of the arguments $2^r v$, $v = \frac{2}{11}\omega_3$, are derivable from each other when considered as elliptic functions of $v + \frac{2}{5}r\omega$, $r = 0, 1, 2, 3, 4$.

The connexion with KLEIN's parameter τ is made through

$$\mathrm{K} = -11\tau, \quad \mathrm{K}'^2 = 4\mathrm{K}^2(\mathrm{K} - 11) + (10\mathrm{K} + 11)^2 = 121\tau'^2 \qquad (4),$$

and the quintic transformation

$$\mathrm{H} = \frac{1 + 4c + 2c^2 - 5c^3 - 2c^4 + c^5}{c^2(1 + c)^2}$$

$$= \frac{x^{11} + 1}{x^2(x^2 + 1)^2(x^3 + 1)}, \quad \text{if} \quad \frac{1 + c}{c} = x + \frac{1}{x}$$

$$\mathrm{H}'^2 = \frac{(2 + 8c + 12c^2 + 9c^3 - c^4 - 3c^5 - c^6)^2\,\mathrm{C}}{c^6(1 + c)^6} \qquad (5),$$

and then

$$\mathrm{H}^2\mathrm{K}^2 - 110\mathrm{HK} - 121(\mathrm{H} + \mathrm{K}) + 1331 = 0 \qquad (6),$$

an elliptic-function addition equation (L.M.S., 25, p. 244; 27, p. 428).

If analogy is to help us in passing from $\mu = 11$ to 19, the ellipsotomic equation $\gamma_{19} = 0$ should be reducible to the relation

$$\mathrm{H}^2\mathrm{K}^2 - 152\mathrm{HK} - 361(\mathrm{H} + \mathrm{K}) = 0 \qquad (7),$$

where, in terms of FRICK's τ and τ' ('Math. Ann.,' 40),

$$\mathrm{K} = -19\tau, \quad \mathrm{K}'^2 = 4\mathrm{K}^3 + (8\mathrm{K} + 19)^2 = 361\tau'^2 \qquad (8),$$

with the addition of the cubic transformations

$$\mathrm{H} = \frac{a^3 - 5a^2 + 2a + 1}{a^2}, \quad \mathrm{H}'^2 = \frac{(a^3 - 2a - 2)^2\{4a^3 + (2a + 1)^2\}}{a^6} \qquad (9),$$

and K, K′ the same functions of b.

This combination of (7) and (9) leads to an equation of the 12th degree between $a = a(u)$ and $b = a\left(u + \frac{2r\omega}{9}\right)$, functions of the elliptic argument

$$u = \int_a^\infty \frac{da}{\sqrt{\{4a^3 + (2a + 1)^2\}}} \qquad (10).$$

The nine division values of the argument $2^{r}v$, $v = \frac{2\omega_3}{19}$ should now be functions of an argument $u + \frac{2r\omega}{9}$, and thence derivable from each other, being grouped in sets of three

$$\left|\begin{matrix} v, & 2^3v, & 2^6v \\ 2v, & 2^4v, & 2^7v \\ 2^2v, & 2^5v, & 2^8v \end{matrix}\right| \quad \text{or} \quad \left|\begin{matrix} v, & 8v, & 7v \\ 2v, & 3v, & 5v \\ 4v, & 6v, & 9v \end{matrix}\right| \tag{11}$$

In passing horizontally in these sets, the substitution connecting $a = a(u)$ and $b = a\left(u + \frac{2r\omega}{3}\right)$ is

$$a^2b^2 - 2ab - a - b = 0 \tag{12}$$

The additional cubic transformation

$$a = \frac{c^3 + 15c^2 + 57c}{(3c + 19)^2} \tag{13}$$

leads to a multiplication relation of the 9th order, connecting

$$\mathrm{H} = \mathrm{H}(u) \quad \text{and} \quad c = \mathrm{H}\left(\tfrac{1}{3}u\right) \tag{14}$$

The relation of the 12th order between a and b is of the 6th order in $p = a + b$ and $q = ab$, representing a C_6 in the coordinates p and q.

But so far the various transformations of $\gamma_{19} = 0$, as given in L.M.S., 25, lead to a C_7 of the fifth degree in each variable, and the reason of this is a mystery still.

17. For $\mu = 21$, applying the trisection equation (8) § 14, with the relations of § 9

$$x = z(1 - z)^2, \quad y = z(1 - z) \tag{1}$$

$$3t^4 + [(1 + z - z^2)^2 - 8z(1 - z)^2]t^3 + 3z^2(1 - z)^3(1 - 3z + z^2)t^2 + 3z^4(1 - z)^6t + z^6(1 - z)^7 = 0 \tag{2}$$

Put

$$t = \frac{z^2(1 - z)^2}{z - w} \tag{3}$$

$$w^3z^2 - (w^4 - 2w^3 + 3w^2 + 0 - 1)z + w(w - 1)^3 = 0 \tag{4}$$

$$z = \frac{w^4 - 2w^3 + 3w^2 + 0 - 1 + \sqrt{\mathrm{W}}}{2w^3} \tag{5}$$

$$z - w = \frac{-w^4 - 2w^3 + 3w^2 + 0 - 1 + \sqrt{\mathrm{W}}}{2w^3} \tag{6}$$

$$\begin{aligned} \mathrm{W} &= w^8 - 8 + 22 - 24 + 11 + 4 - 6 + 0 + 1 \\ &= (w^2 - w + 1)[(w^3 - 3w^2 + 0 + 1)^2 - (w^2 - w)(w^3 - 3w^2 + 0 + 1) + (w^2 - w)^2] \end{aligned} \tag{7}$$

The substitution $\left(w, \frac{1}{1-w}, \frac{w-1}{w}\right)$ gives $\left(z, \frac{1}{1-z}, \frac{z-1}{z}\right)$, and

$$12\wp 3v = -(1+z-z^2)^2 - 4z(1-z)^2 \qquad (8),$$

$$12\wp 6v = -(1+z-z^2)^2 + 8z(1-z)^2 \qquad (9),$$

$$s(3v) = -z(1-z)^2, \quad is'(3v) = z(1-z)^2 \qquad (10),$$

$$s(6v) = 0, \qquad is'(6v) = z^2(1-z)^3 \qquad (11),$$

$$s(9v) = z^2(1-z), \qquad is'(9v) = z^3(1-z) \qquad (12),$$

$$s(7v) = -\frac{z^2(1-z)^2}{z-w}, \quad is'(7v) = \sqrt{\frac{z^4(1-z)^6}{(z-t)^3}\left[4w^2 + (z-w)\left(w+\frac{1}{1-z}\right)^2\right]} \qquad (13).$$

18. For $\mu = 23$ the class of the modular equation is 2, so that simple relations cannot be anticipated; but as the class is zero for $\mu = 25$, it is possible that the ellipsotomic equation $\gamma_{25} = 0$ may be susceptible of reduction (L.M.S., 25, p. 275).

19. As mechanical applications of the preceding integrals of the First Stage we may cite the case of LÉVY's "Elastica," mentioned in § 5 and discussed in the 'Math. Ann.,' 52, the Spherical Catenary (L.M.S., 27), and the Velarium surfaces considered in F. KÖTTER's 'Inaugural-Dissertation' (Halle, 1883).

Take an umbrella with straight ribs, and hold its axis vertical, as an illustration of a velarium.

If the gore laid out flat forms a sector of a circle, then it is obvious that for any other angle between the radii formed by the ribs, the edge and its concentric lines form spherical catenary curves, as shown by KÖTTER's equations.

But with triangular gores (KÖTTER, 'Diss.,' p. 38) the projection of the edge on a horizontal plane is given by

$$\theta = \tfrac{1}{2}\int \frac{\mathrm{A}\,dt}{t\sqrt{\{(\mathrm{B}-\frac{1}{2}t)^2 t - \mathrm{A}^2(1-t)\}}} \qquad (1),$$

with $t = \left(\frac{r}{a}\right)^2$; and this is reducible immediately to our standard form (1), § 5, by putting

$$t = \frac{s}{\mathrm{M}^2} \qquad (2),$$

$$\begin{aligned} \mathrm{S} &= 4s(s+x)^2 - \{(y+1)s+xy\}^2 \\ &= 16\mathrm{M}^6\mathrm{T} = 4\mathrm{M}^2 t(\mathrm{M}^2 t - 2\mathrm{M}^2\mathrm{B})^2 + 16\mathrm{M}^4\mathrm{A}^2(\mathrm{M}^2 t - \mathrm{M}^2) \\ &= 4s(s-2\mathrm{M}^2\mathrm{B})^2 + 16\mathrm{M}^4\mathrm{A}^2(s-\mathrm{M}^2) \end{aligned} \qquad (3),$$

and equating coefficients,

$$16\mathrm{M}^2\mathrm{B} = (1 + y)^2 - 8x \qquad (4),$$

$$8\mathrm{M}^4(\mathrm{A}^2 + \mathrm{B}^2) = 2x^2 - xy(1 + y) \qquad (5),$$

$$4\mathrm{A}\mathrm{M}^3 = xy \qquad (6),$$

so that

$$256\mathrm{M}^4\mathrm{A}^2 = 64x^2 - 32xy(1 + y) - [(1 + y)^2 - 8x]^2$$
$$= (1 + y)[16x(1 - y) - (1 + y)^3] \qquad (7),$$

$$\mathrm{M}^2 = \frac{16x^2y^2}{(1 + y)[16x(1 - y) - (1 + y)^3]} \qquad (8).$$

Now denoting the elliptic argument by u, where

$$u = \int \frac{ds}{\sqrt{\mathrm{S}}} \qquad (9),$$

$$\theta + u\mathrm{P}(2v) = \mathrm{I}(2v) \qquad (10),$$

and the preceding integrals can be utilised for the construction of solvable cases.

The chief interest is in the purely algebraical case, obtained by putting $\mathrm{P}(2v) = 0$.

Thus we find for $\mu = 5$, putting $y = x = \frac{1}{3}$, in (3), § 7,

$$2r^{\frac{5}{2}} \cos \tfrac{5}{2}\theta = (r + a)\sqrt{(2r^3 - 4ar^2 + 6a^2r + 3a^3)} \qquad (11),$$

$$2r^{\frac{5}{2}} \sin \tfrac{5}{2}\theta = (r - a)\sqrt{(2r^3 + 4ar^2 + 6a^2r + 3a^3)} \qquad (12).$$

20. The expression of the pseudo-elliptic spherical catenary, discussed in L.M.S., 27, p. 127, can be halved in degree by changing to the stereographic projection, with

$$\tan \tfrac{1}{2}\theta = t, \quad z = \cos\theta = \frac{1 - t^2}{1 + t^2} \qquad (1);$$

$$\mathrm{Z} = (1 - z^2)(h - z)^2 - \mathrm{A}^2 = \frac{\mathrm{A}^2\mathrm{T}}{(1 + t^2)^4} \qquad (2),$$

$$\mathrm{T} = 4t^2\left(\frac{h - 1}{\mathrm{A}} + \frac{h + 1}{\mathrm{A}}t^2\right)^2 - (1 + t^2)^4 = \mathrm{T}_1\mathrm{T}_2 \qquad (3),$$

$$\mathrm{T}_1, \mathrm{T}_2 = 2t\left(\frac{h - 1}{\mathrm{A}} + \frac{h + 1}{\mathrm{A}}t^2\right) \mp (1 + t^2)^2 \qquad (4),$$

$$\psi = \int \frac{\mathrm{A}\,dz}{(1 - z^2)\sqrt{\mathrm{Z}}} = -\tfrac{1}{2}\int \frac{(1 + t^2)^2\,dt^2}{t^2\sqrt{\mathrm{T}_1\mathrm{T}_2}} \qquad (5).$$

In a pseudo-elliptic case, with

$$\mu = 2n + 1, \quad v = \frac{2\omega_3}{\mu}, \quad \mathrm{M}^2 = -\frac{y + 1}{2x}, \quad \frac{\mathrm{A}}{\mathrm{M}} = y + 1, \quad \mathrm{A}^2 - h^2 = 2y + 1 \qquad (6),$$

we put

$$\psi + \frac{4\mathrm{P}(r)}{y+1}\int \frac{dt^2}{\sqrt{\mathrm{T}_1\mathrm{T}_2}} = \chi \qquad (7),$$

and now the expression for the catenary can be reduced to

$$\begin{aligned} \mathrm{N}\,(\tan\tfrac{1}{2}\,\theta e^{\chi i})^{n+\frac{1}{2}} = {} & (\mathrm{B} + \mathrm{B}_1 t + \mathrm{B}_2 t^2 + \ldots + \mathrm{B}_{2n-1}t^{2n-1})\sqrt{\mathrm{T}_1} \\ & + i\,(\mathrm{B} - \mathrm{B}_1 t + \mathrm{B}_2 t^2 - \ldots - \mathrm{B}_{2n-1}t^{2n-1})\sqrt{\mathrm{T}_2} \qquad (8). \end{aligned}$$

Thus for

$$\mu = 7, \quad x = z(1-z)^2, \quad y = z(1-z) \qquad (9),$$

$$\mathrm{P}(r) = \frac{5 - z - z^2}{14} = 0 \quad \text{for} \quad z = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{21} \qquad (10);$$

and we can calculate M, A, h, and the six B's in

$$\begin{aligned} \mathrm{N}\,(\tan\tfrac{1}{2}\,\theta e^{\psi i})^{\frac{7}{2}} = {} & (\mathrm{B} + \mathrm{B}_1 t + \ldots + \mathrm{B}_5 t^5)\sqrt{\mathrm{T}_1} \\ & + i\,(\mathrm{B} - \mathrm{B}_1 t + \ldots - \mathrm{B}_5 t^5)\sqrt{\mathrm{T}_2} \qquad (11); \end{aligned}$$

this catenary has been drawn stereoscopically by the late Mr. T. I. DEWAR.

21. ABEL's integrals are applicable immediately to the construction in polar co-ordinates r and θ of algebraical orbits or catenaries under a central attraction of the form (HUGO GYLDÉN, 'Kongl. Svenska Vetenskaps-Akademiens Handlingar,' 1879)

$$\mathrm{P} = \frac{\mu}{r^2} + \mu_0 + \mu_1 r \qquad (1);$$

with $\mu = 0$ the orbit can be realised by two balls, connected by an elastic thread, whirling round each other in the air; and the addition of a term to P, varying inversely as r^3, merely has the effect of qualifying the angle θ by a factor.

Putting $r = \frac{1}{u}$, and denoting the velocity in the orbit by v and twice the rate of description of area by h,

$$\begin{aligned} v^2 = h^2\left(\frac{du^2}{d\theta^2} + u^2\right) &= \mathrm{H} - 2\int \mathrm{P}\,dr \\ &= 2\frac{\mu}{r} + \mathrm{H} - 2\mu_0 r - \mu_1 r^2 \qquad (2), \end{aligned}$$

$$u^2\left(\frac{du}{d\theta}\right)^2 = -\mu^4 + 2\mathrm{A}u^3 + \mathrm{B}u^2 - 2\mathrm{C}u - \mathrm{D} = \mathrm{U} \qquad (3),$$

$$\mathrm{A} = \frac{\mu}{h^2}, \quad \mathrm{B} = \frac{\mathrm{H}}{h^2}, \quad \mathrm{C} = \frac{\mu_0}{h^2}, \quad \mathrm{D} = \frac{\mu_1}{h^2} \qquad (4),$$

$$\theta = \int \frac{u\,du}{\sqrt{\mathrm{U}}} \qquad (5),$$

and now put $u = z - k$ to identify with ABEL's results.

Thus for $\mu = 3$ in § 6 (5), putting $u = z + \frac{1}{3}a$, and $a = 3b$,

$$\theta = \frac{2}{3} \cos^{-1} (u + b) \sqrt{\frac{u - b}{p}} = \frac{2}{3} \sin^{-1} \sqrt{\left\{1 - \frac{(u + b)^2 (u - b)}{p}\right\}} \qquad (6),$$

$$\text{A} = - b, \quad \text{B} = 3b^2, \quad 2\text{C} = - 4b^3 - p, \quad \text{D} = b (4b^3 + p) \qquad (7).$$

For instance, $b = 0$ makes $\mu = 0$, $\text{H} = 0$, $\mu_1 = 0$, and the orbit is

$$r^{\frac{3}{2}} \cos \tfrac{3}{2} \theta = c^{\frac{3}{2}} \qquad (8),$$

described under a constant central force.

For $\mu = 5$ in (7), § 7, putting

$$w = 5z - 1 - 2x \qquad (9),$$

$$\theta = \frac{2}{5} \cos^{-1} \frac{w^2 - (8 + x) w + 2 (8 + x) (1 - 3x)}{50x \sqrt{5}} \sqrt{(w + 1 - 3x)}$$

$$= \frac{2}{5} \sin^{-1} \frac{w - 4 - 3x}{50x \sqrt{5}} \sqrt{\{- w^3 + (7 - x) w^2 + 8 (- 1 + 11x + x^2) w + 4 (4 + 3x) (- 1 + 11x + x^2)\}} \qquad (10),$$

and to make $\mu = 0$, put $x = - 3$, so that with $w = 5u$,

$$\theta = \frac{2}{5} \cos^{-1} \frac{u^2 - u + 4}{6} \sqrt{(u + 2)}$$

$$= \frac{2}{5} \sin^{-1} \frac{u + 1}{6} \sqrt{(- u^3 + 2u^2 - 8u + 4)} \qquad (11),$$

an orbit described by a body attached to an elastic thread, which is led through a fixed origin, which can be written

$$6r^{\frac{5}{2}} \cos \tfrac{5}{2} \theta = (4r^2 - ar + a^2) \sqrt{(2r + a)} \qquad (12),$$

$$6r^{\frac{5}{2}} \sin \tfrac{5}{2} \theta = (r + a) \sqrt{(4r^3 - 8ar^2 + 2a^2r - a^3)} \qquad (13).$$

So also for $\mu = 7$.

The Second Stage.

22. But in the dynamical applications, such as POINSOT's herpolhode and JACOBI's associated motion of the top, the integral of the Second Stage is required, corresponding to an even value of μ, and S can now be resolved into its factors

$$\text{S} = 4 (s - s_1) (s - s_2) (s - s_3) \qquad (1),$$

while in most cases of the applications

$$s_1 > \sigma > s_2 > s > s_3 \qquad (2),$$

so that

$$v = \omega_1 + f\omega_3, \quad u = f'\omega_1 + \omega_3 \tag{3}$$

where f, f' denotes real fractions.

To obtain this resolution of S we find it convenient to put (L.M.S., 27, p. 449),

$$x = -\frac{m^3\alpha}{(\alpha - m)^2}, \quad y = -\frac{(1-2m)\alpha}{\alpha - m} \tag{4}$$

and with $s = t^2$,

$$\mathrm{T}_1, \mathrm{T}_2 = \left(t \pm \frac{m\alpha}{\alpha - m}\right)\left\{2t^2 \mp \frac{mt}{\alpha - m} + \frac{m^2(1-2m)\alpha}{(\alpha - m)^2}\right\} \tag{5}$$

Denoting the roots of $\mathrm{S} = 0$, irrespective of order of magnitude, by a^2, b^2, c^2, we can put

$$a, b = \frac{m}{4(\alpha - m)}[1 \pm \sqrt{\{1 - 8(1-2m)\alpha\}}], \quad c = \frac{m\alpha}{\alpha - m} \tag{6}$$

$$a + b = \frac{m}{2(\alpha - m)}, \quad a - b = \frac{m\sqrt{\{1 - 8(1-2m)\alpha\}}}{2(\alpha - m)} \tag{7}$$

$$(a + c)(b + c) = \frac{m\alpha(\alpha - m + 1)}{(\alpha - m)^2}, \quad (a - c)(b - c) = \frac{m^3\alpha}{\alpha - m} \tag{8}$$

With $\frac{1}{2}\mu v$ congruent to a half-period ω, we can take

$$s(\tfrac{1}{2}\mu v) = s(\omega) = c^2 = \frac{m^2\alpha^2}{(\alpha - m)^2}, \quad \mathrm{S}(\tfrac{1}{2}\mu v) = 0 \tag{9}$$

and this leads to a relation between α and m, by means of which they are expressible theoretically in terms of a single variable.

Also, as shown in L.M.S., 27, p. 450,

$$s(\omega) - s(v) = \frac{m^2\alpha}{\alpha - m} \tag{10}$$

$$s(\omega) - s(2v) = \frac{m^2\alpha^2}{(\alpha - m)^2} \tag{11}$$

$$s(\omega) - s(3v) = \frac{(1-m)^2\alpha}{\alpha - m} \tag{12}$$

$$s(\omega) - s(4v) = \frac{m^2\{(1-2m)\alpha - m(1-m)\}^2}{(1-2m)^2(\alpha - m)^2} \tag{13}$$

$$\begin{aligned} s(\omega) - s(5v) &= \frac{m^2\alpha}{\alpha - m}\left\{\frac{(1-2m)\alpha - m^2(1-m)}{(1-2m)\alpha - m(1-m)^2}\right\}^2 \\ &= \frac{m^2\alpha}{\alpha - m}\left(\frac{\mathrm{N}_5}{\mathrm{D}_5}\right)^2 \end{aligned} \tag{14}$$

where

$$D_5 = (1 - 2m)\,\alpha - m\,(1 - m)^2,$$

and N_5 is obtained from D_5 by a change of m into $1 - m$, and α into $- \alpha$.

$$s\,(\omega) - s\,(6v) = \frac{\alpha^2}{(\alpha - m)^2}\left(\frac{N_6}{D_6}\right)^2 \qquad (15),$$

$$D_6 = 2\,(1 - m)\,(1 - 2m)\,\alpha - m\,(1 - m)^2,$$

$$N_6 = (1 - 2m + 2m^2)\,(1 - 2m)\,\alpha - m\,(1 - m)\,(1 - 3m + 3m^2)\,;$$

$$s\,(\omega) - s\,(7v) = \frac{m^2\alpha}{\alpha - m}\left(\frac{N_7}{D_7}\right)^2 \qquad (16),$$

$$D_7 = (1 - 2m)^3\,\alpha^2 - m\,(1 - m)\,(1 - 2m)\,(1 - 3m)\,\alpha - m^4\,(1 - m)^2,$$

and N_7 is obtained by a change of m into $1 - m$ and α into $- \alpha$;

$$s\,(\omega) - s\,(8v) = \frac{m^2}{(1 - 2m)^2\,(\alpha - m)^2}\left(\frac{N_8}{D_8}\right)^2 \qquad (17),$$

$$D_8 = \{(1 - 2m)\,\alpha - m\,(1 - m)\}\,\{(1 - 2m)\,\alpha - m\,(1 - m)\,(1 - 2m + 2m^2)\},$$

$$\begin{aligned} N_8 = (1 - 2m)^3\,\alpha^3 - m\,(1 - m)\,(1 - 2m)^2\,(1 + 2m - 2m^2)\,\alpha^2 \\ + 4m^3\,(1 - m)^3\,(1 - 2m)\,\alpha - m^4\,(1 - m)^4\,; \end{aligned}$$

$$s\,(\omega) - s\,(9v) = \frac{(1 - m)^2\,\alpha}{\alpha - m}\left(\frac{N_9}{D_9}\right)^2 \qquad (18),$$

$$\begin{aligned} D_9 = (1 - 2m)^3\,(3 - 6m + 4m^2)\,\alpha^3 - 2m\,(1 - m)\,(1 - 2m)^2\,(3 - 7m + 5m^2)\,\alpha^2 \\ + m^2\,(1 - m)^2\,(1 - 2m)\,(4 - 10m + 7m^2)\,\alpha - m^3\,(1 - m)^6, \end{aligned}$$

and a change of α into $- \alpha$ and m into $1 - m$ gives N_9;

$$s\,(\omega) - s\,(10v) = \frac{m^2\alpha^2}{(\alpha - m)^2}\left(\frac{N_{10}}{D_{10}}\right)^2 \qquad (19),$$

$$\begin{aligned} D_{10} = \{(1 - 2m)\,\alpha - m\,(1 - m)^2\}\,\{(1 - 2m)\,\alpha - m^2\,(1 - m)\} \\ \{(1 - 2m)^4\,\alpha^2 - m\,(1 - m)\,(1 - 2m)\,(1 - 6m + 6m^2)\,\alpha - m^3\,(1 - m^3)\}, \end{aligned}$$

$$\begin{aligned} N_{10} = (1 - 2m)^6\,\alpha^4 - 2m\,(1 - m)\,(1 - 2m)^3\,(2 - 7m + 7m^2)\,\alpha^3 \\ + 2m^2\,(1 - m)^2\,(1 - 2m)^2\,(3 - 11m + 13m^2 - 4m^3 + 2m^4)\,\alpha^2 \\ - m^3\,(1 - m)^3\,(1 - 2m)\,(4 - 17m + 27m^2 - 20m^3 + 10m^4)\,\alpha \\ + m^4\,(1 - m)^4\,(1 - 5m + 10m^2 - 10m^3 + 5m^4). \end{aligned}$$

A simplification is effected by putting

$$(1 - 2m)\,\alpha = m\,(1 - m)\,\beta, \quad \text{and} \quad 1 - 2m = p \qquad (20),$$

and now the change of m into $1 - m$ and α into $-\alpha$ is equivalent to a change of sign of p, leaving β unchanged.

Thus we find, with $2\beta - 1 = \epsilon$,

$$s(\omega) - s(11v) = \frac{m^2\alpha}{\alpha - m}\left(\frac{N_{11}}{D_{11}}\right)^2 \qquad (21),$$

$$D_{11} = -p^5 + \epsilon(\epsilon-2)(\epsilon^2-\epsilon-1)p^4 + \epsilon(\epsilon^2+1)p^3 + \epsilon^2(\epsilon^2-3)p^2 - \epsilon^3(\epsilon^2-\epsilon+1)p + \epsilon^3,$$

$$N_{11} = \quad p^5 + \epsilon(\epsilon-2)(\epsilon^2-\epsilon-1)p^4 - \epsilon(\epsilon^2+1)p^3 + \epsilon^2(\epsilon^2-3)p^2 + \epsilon^3(\epsilon^2-\epsilon+1)p + \epsilon^3,$$

$$\begin{aligned} N_{11} - D_{11} &= 2p\{p^4 - \epsilon(\epsilon^2+1)p^2 + \epsilon^3(\epsilon^2-\epsilon+1)\}, \\ &= 2p(p^2-\epsilon^2)(p^2-\epsilon^3+\epsilon^2-\epsilon), \end{aligned}$$

$$\begin{aligned} N_{11} + D_{11} &= 2\epsilon\{(\epsilon-2)(\epsilon^2-\epsilon-1)p^4 + \epsilon(\epsilon^2-3)p^2 + \epsilon^2\}, \\ &= 2\epsilon\{(\epsilon-2)p^2+\epsilon\}\{(\epsilon^2-\epsilon-1)p^2+\epsilon\}. \end{aligned}$$

Thus, if $\mu = 22$, $s(11v) = s(\omega)$, and $N_{11} = 0$, which replaces in a much simpler form the relation $\gamma_{22} = 0$.

So also $D_{11} = 0$ makes $s(11v) = \infty$, $v = \frac{2\omega}{11}$, and so enables us to connect up the results for $\mu = 11$; and other values of μ can be treated in the same way.

23. There are three cases to consider, relative to the half-period ω, to which $\frac{1}{2}\mu v$ is congruent:—

I. $$\omega = \omega_1, \quad s(\tfrac{1}{2}\mu v) = s(\omega_1) = s_1 = c^2, \quad s_2 = a^2, \quad s_3 = b^2 \qquad (1),$$

and introducing WEBER's function $f\omega$, or KIEPERT's equivalent function $L(2)$,

$$L(2)^{24} = (f_1\omega)^{24} = 16\frac{\kappa^4}{\kappa'^2} = \frac{16(s_2-s_3)^2}{(s_1-s_3)(s_1-s_2)} = \frac{1-8(1-2m)\alpha}{\alpha^2(\alpha-m)(\alpha-m+1)} \qquad (2),$$

and to the complementary modulus κ',

$$\frac{1}{\mathrm{cn}^2 f\mathrm{K}'}, \quad \frac{1}{\mathrm{dn}^2 f\mathrm{K}'} = \frac{1 + 4\alpha \pm \surd\{1 - 8(1-2m)\alpha}{2} \qquad (3),$$

$$\mathrm{cn}\,2f\mathrm{K}' = \frac{b}{c}, \quad \mathrm{dn}\,2f\mathrm{K}' = \frac{a}{c}, \quad \mathrm{sn}\,(1-2f)\,\mathrm{K}' = \frac{b}{a} \qquad (4);$$

II. $$\omega = \omega_2, \quad s(\tfrac{1}{2}\mu v) = s(\omega_2) = s_2 = c^2, \quad s_1 = a^2, \quad s_3 = b^2 \qquad (5),$$

$$(f\omega)^{24} = \frac{16}{\kappa^2\kappa'^2} = \frac{16(s_1-s_3)^2}{(s_2-s_3)(s_1-s_2)} = \frac{-1+8(1-2m)\alpha}{\alpha^2(\alpha-m)(\alpha-m+1)} = -L(2)^{24} \qquad (6),$$

$$\frac{1}{\mathrm{sn}^2 f\mathrm{K}'}, \quad \mathrm{dn}^2(1-f)\,\mathrm{K}' = \frac{1 + 4\alpha \pm \surd\{1 - 8(1-2m)\alpha\}}{2} \qquad (7),$$

$$\mathrm{cn}\,2f\mathrm{K}' = \frac{a}{b}, \quad \mathrm{dn}\,2f\mathrm{K}' = \frac{c}{a}, \quad \mathrm{sn}\,(1-2f)\,\mathrm{K}' = \frac{b}{c} \qquad (8);$$

III. $$\omega = \omega_3, \quad s\left(\tfrac{1}{2}\mu v\right) = s(\omega_3) = s_3 = c^2, \quad s_1 = a^2, \quad s_2 = b^2 \qquad (9),$$

$$(f_2\omega)^{24} = 16\frac{\kappa'^4}{\kappa^2} = 16\frac{(s_1 - s_2)^2}{(s_1 - s_3)(s_2 - s_3)} = \frac{1 - 8(1 - 2m)\alpha}{\alpha^2(\alpha - m)(\alpha - m + 1)} = \mathrm{L}(2)^{24} \qquad (10),$$

$$\frac{1}{\kappa'^2 \mathrm{sn}^2 f\mathrm{K}'}, - \frac{\kappa^2}{\kappa'^2 \mathrm{cn}^2 f\mathrm{K}'} = \frac{1 - 4\alpha \pm \surd\{1 - 8(1 - 2m)\alpha\}}{2}. \qquad (11).$$

$$\mathrm{cn}\,2f\mathrm{K}' = \frac{c}{a}, \quad \mathrm{dn}\,2f\mathrm{K}' = \frac{b}{a}, \quad \mathrm{sn}\,(1 - 2f)\,\mathrm{K}' = \frac{c}{b} \qquad (12).$$

A fourth case, where $1 - 8(1 - 2m)\alpha$ is negative and a, b imaginary, need not be discussed here.

The Weierstrassian Sigma Functions of the Division-values (Theilwerthe) can now be expressed in terms of the Jacobian Theta and Eta Functions by the relations on p. 52 of SCHWARZ, 'Formeln der elliptischen Functionen,' so that

$$\mu = 4n + 2, \quad v = \mathrm{K} + \frac{\mathrm{K}'i}{2n + 1} \qquad (13),$$

$$\frac{\Theta\dfrac{\mathrm{K}'}{2n + 1}}{\Theta\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_2)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{1}{4n+2}} \qquad (14),$$

$$\frac{\mathrm{H}\dfrac{2\mathrm{K}'}{2n + 1}}{\Theta\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_2)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{4}{4n+2}} \qquad (15),$$

$$\frac{\Theta\dfrac{2\mathrm{K}'}{2n + 1}}{\Theta 0} = \surd(s_1 - s_3)\, x^{-\frac{1}{3}}\lambda^{-\frac{4}{4n+2}}\,\mathrm{ns}\,\frac{2\mathrm{K}'}{2n + 1}$$

$$= x^{-\frac{1}{3}}\lambda^{-\frac{4}{4n+2}}\,\frac{ma}{\alpha - m} \qquad (16).$$

$$\mu = 4n, \quad v = \mathrm{K} + \frac{\mathrm{K}'i}{2n} \qquad (17),$$

$$\frac{\Theta\dfrac{\mathrm{K}'}{2n}}{\mathrm{H}\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_3)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{1}{4n}} \qquad (18),$$

$$\frac{\mathrm{H}\dfrac{\mathrm{K}'}{n}}{\mathrm{H}\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_3)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{1}{n}} \qquad (19),$$

$$\frac{\Theta\dfrac{\mathrm{K}'}{n}}{\Theta 0} = x^{-\frac{1}{3}}\lambda^{-\frac{1}{n}}\,\frac{m\alpha}{\alpha - m} \qquad (20).$$

24. When μ is of the form $4n + 2$, so that, as required in dynamical problems,

$$v = \omega_1 + \frac{r\omega_3}{2n + 1}, \quad \text{or} \quad \mathrm{K} + \frac{r\mathrm{K}'i}{2n + 1} \tag{1},$$

then $\frac{1}{2}\mu v = (2n + 1)v$ is congruent to ω_1 or ω_2, according as r is even or odd; and the integral $\mathrm{I}(v)$ is expressible in the form

$$\begin{aligned}\mathrm{I}(v) &= \int_{b^2}^{s} \frac{-\mathrm{P}(v)(\sigma - s) + \frac{1}{2}x}{\sigma - s} \frac{ds}{\sqrt{\mathrm{S}}} \\ &= \frac{1}{2n + 1} \cos^{-1} \frac{s^n + \mathrm{P}_1 s^{n-1} + \ldots + \mathrm{P}_n}{(\sigma - s)^{n+\frac{1}{2}}} \sqrt{(c^2 - s)} \\ &= \frac{1}{2n + 1} \sin^{-1} \frac{\mathrm{Q}_1 s^{n-1} + \ldots + \mathrm{Q}_n}{(\sigma - s)^{n+\frac{1}{2}}} \sqrt{(a^2 - s \,.\, s - b^2)}\end{aligned} \tag{2},$$

and when the quantities α, m and therefore a, b, c, σ, x, and $\mathrm{P}(v)$ have been determined as algebraical functions of a parameter, the calculation of $\mathrm{P}_1, \ldots \mathrm{P}_n$, $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$, is effected readily by a differentiation and verification.

It will be found that

$$\mathrm{Q}_1 = (2n + 1)\,\mathrm{P}(v) \tag{3},$$

also

$$\frac{\mathrm{P}(\omega_1 + f\omega_3)}{\sqrt{(s_1 - s_3)}} = \mathrm{zn}(f\mathrm{K}', \kappa'), \quad \frac{\mathrm{P}(f\omega_3)}{\sqrt{(s_1 - s_3)}} = \mathrm{zs}f\mathrm{K}' \tag{4},$$

$\mathrm{zn}u$ denoting Jacobi's Zeta function $\mathrm{Z}u$ or $\frac{d}{du}\log \Theta u$, while

$$\mathrm{zs}u = \mathrm{zn}u + \frac{d}{du}\log \mathrm{sn}u = \frac{d}{du}\log \mathrm{H}u. \tag{5}$$

in Glaisher's notation; also

$$\mathrm{P}(2n + 1)v = 0, \quad \mathrm{P}(v) + \mathrm{P}(2nv) = \frac{-\frac{1}{2}x}{\sigma - s(\omega)} = -\frac{1}{2}\frac{m}{\alpha - m} \tag{6}.$$

Here already the degree of the results obtained by Abel's method have been halved in (2); and the degree can be halved again by the substitution

$$y^2 = \frac{\sigma - s}{c^2 - s} \tag{7},$$

and now

$$\begin{aligned}\mathrm{I}(v) &= \frac{1}{2n + 1} \cos^{-1} \frac{s^n + \mathrm{P}_1 s^{n-1} + \ldots + \mathrm{P}_n}{(\sigma - s)^n y} \\ &= \frac{1}{2n + 1} \cos^{-1} \frac{\mathrm{B}_0 y^{2n} + \mathrm{B}_1 y^{2n-2} + \ldots + \mathrm{B}_n}{y^{2n+1}} \\ &= \frac{2}{2n + 1} \cos^{-1} \frac{y^{n-1} + \mathrm{A}_1 y^{n-2} + \ldots + \mathrm{A}_{n-1}}{y^{n+\frac{1}{2}}} \sqrt{\{\tfrac{1}{2}(y+1)(y^2+\mathrm{C}_1 y+\mathrm{C}_2)\}} \\ &= \frac{2}{2n + 1} \sin^{-1} \frac{y^{n-1} - \mathrm{A}_1 y^{n-2} + \ldots + (-1)^{n-1}\mathrm{A}_{n-1}}{y^{n+\frac{1}{2}}} \sqrt{\{\tfrac{1}{2}(y-1)(y^2-\mathrm{C}_1 y+\mathrm{C}_2)\}}\end{aligned} \tag{8}.$$

Denoting by a'^2, b'^2 the values of y^2 corresponding to a^2, b^2 of s,

$$a'^2 = \frac{\sigma - a^2}{c^2 - a^2}, \quad b'^2 = \frac{\sigma - b^2}{c^2 - b^2} \tag{9}$$

reducing to

$$a'^2,\ b'^2 = \frac{2}{1 + 4\alpha \pm \sqrt{\{1 - 8(1 - 2m)\alpha\}}} \tag{10}$$

so that

$$a' = \operatorname{cn} f\mathrm{K}', \quad b' = \operatorname{dn} f\mathrm{K}'. \qquad \text{in Region I.} \tag{11}$$

$$a' = \operatorname{sn} f\mathrm{K}', \quad b' = \frac{1}{\operatorname{dn}(1 - f)\mathrm{K}'}, \text{ in Region II.} \tag{12}$$

Changing the variable to y in the integral

$$\mathrm{I}(v') = \int_{1, \text{ or } b'}^{y} \frac{\mathrm{P}(v')\, y^2 - \mathrm{Q}(v')}{y^2} \frac{dy^2}{\sqrt{\mathrm{Y}}} \tag{13}$$

where

$$\mathrm{Y} = 4(y^2 - 1)(y^2 - a'^2)(y^2 - b'^2) \tag{14}$$

$$\mathrm{P}(v') = \frac{\mathrm{P}(\omega - v)}{\mathrm{M}'}, \quad \mathrm{M}'^2 = \frac{m^2\alpha(\alpha - m + 1)}{(\alpha - m)^2} \tag{15}$$

$$\mathrm{Q}(v') = \frac{\frac{1}{2}is'(\omega - v)}{\mathrm{M}'^3} = a'b' = \mathrm{C}_2 \tag{16}$$

$$\mathrm{C}_2^{\,2} = a'^2 b'^2 = \frac{1}{4\alpha(\alpha - m + 1)} \tag{17}$$

$$\mathrm{C}_1^{\,2} - 2\mathrm{C}_2 = a'^2 + b'^2 = \frac{1 + 4\alpha}{4\alpha(\alpha - m + 1)} \tag{18}$$

$$\kappa^2 = \frac{1 - a'^2}{b'^2 - a'^2}, \quad \kappa'^2 = \frac{b'^2 - 1}{b'^2 - a'^2} \tag{19}$$

$$\kappa^2 - \kappa'^2 = \frac{-1 + 4(1 - 2m)\alpha + 8\alpha^2}{\sqrt{1 - 8(1 - 2m)\alpha}} \tag{20}$$

25. To connect up ABEL's results for even values of μ we take $n = 1$ in (9), § 5, so that

$$a = \frac{2x - y - y^2}{y}, \quad b = \frac{-x(y - x - y^2)}{y^2}, \quad p = 4xy \tag{1}$$

$$\begin{aligned} \mathrm{Z} &= -(z^2 - az + b)^2 + pz \\ &= -(z^2 - az + b + 2s)^2 + 4(\mathrm{A}z + \mathrm{B})^2 \end{aligned} \tag{2}$$

where

$$\mathrm{A}^2 = s, \quad 2\mathrm{AB} = -as + xy, \quad \mathrm{B}^2 = s(s + b) \tag{3}$$

and therefore the resolving cubic becomes

$$4s^2(s + b) - (-as + xy)^2 = 0 \tag{4}$$

or

$$S = 4s(s + x)^2 - [(1 + y)s + xy]^2 = 0 \qquad (5).$$

Then, with

$$x = -\frac{m^3\alpha}{(\alpha - m)^2}, \quad y = -\frac{(1 - 2m)\alpha}{\alpha - m}, \quad s = \frac{m^2\alpha^2}{(\alpha - m)^2} \qquad (6),$$

$$Az + B = \frac{m\alpha}{\alpha - m}\left[z + \frac{m(1 - 2m)\alpha - m^2(1 - m)}{(1 - 2m)(\alpha - m)}\right] \qquad (7),$$

and Z breaks up into two quadratic factors, Z_1 and Z_2,

$$\begin{aligned} Z_1 &= z^2 - \frac{m(1 - 2m + 2m^2)}{(1 - 2m)(\alpha - m)}z + \frac{m^7(1 - m)^2}{(1 - 2m)^2(\alpha - m)^2} \\ &= \left[z - \frac{m^3}{(1 - 2m)(\alpha - m)}\right]\left[z - \frac{m(1 - m)^2}{(1 - 2m)(\alpha - m)}\right] \qquad (8), \end{aligned}$$

$$Z_2 = z^2 + \frac{m\{4(1 - 2m)\alpha - 1 + 2m - 2m^2\}}{(1 - 2m)(\alpha - m)}z + \frac{m^2[2(1 - 2m)\alpha - m(1 - m)]^2}{(1 - 2m)^2(\alpha - m)^2} \qquad (9);$$

$$Z_1 = \left[z - \frac{m^2(1 - m)}{(1 - 2m)(\alpha - m)}\right]^2 - \frac{m(1 - 2m)}{\alpha - m}z \qquad (10),$$

$$Z_2 = \left[z + \frac{2m(1 - 2m)\alpha - m^2(1 - m)}{(1 - 2m)(\alpha - m)}\right]^2 - \frac{m(1 - 2m)}{\alpha - m}z \qquad (11);$$

and by means of a homogeneity factor

$$M = \frac{(1 - 2m)(\alpha - m)}{m} \qquad (12),$$

replacing z by $\frac{z}{M}$, we may replace Z_1 and Z_2 by

$$\begin{aligned} Z_1 &= \{z - m(1 - m)\}^2 - (1 - 2m)^2 z \\ &= (z - m^2)\{z - (1 - m)^2\} \qquad (13), \end{aligned}$$

$$Z_2 = \{z + 2(1 - 2m)\alpha - m(1 - m)\}^2 - (1 - 2m)^2 z \qquad (14).$$

26. The resolution of T in the spherical catenary for $\mu = 4n + 2$ is not the same as for $\mu = 2n + 1$; we must put

$$T_1, T_2 = \mp(1 + 2\lambda t + t^2)^2 + 2t(\mu + \nu t^2) \qquad (1),$$

and then in § 20

$$\mu^2 = \lambda - 1 + \left(\frac{h - 1}{A}\right)^2, \quad 2\mu\nu = \lambda^2 - 1 + 2\frac{h^2 - 1}{A^2}, \quad \nu^2 = \lambda - 1 + \left(\frac{h + 1}{A}\right)^2 \qquad (2);$$

so that

$$\left(\lambda^2 - 1 + 2\frac{h^2 - 1}{A^2}\right)^2 - 4\left\{\lambda - 1 + \left(\frac{h - 1}{A}\right)^2\right\}\left\{\lambda - 1 + \left(\frac{h + 1}{A}\right)^2\right\} = 0 \qquad (3).$$

a quartic for λ, having a root $\lambda = 1$, which was used for $\mu = 2n + 1$; the remaining cubic, putting $\lambda = 2\mathrm{B} - 1$, becomes

$$\mathrm{B}^3 - \mathrm{B}^2 + \left(\frac{h^2 - 1}{\mathrm{A}^2} - 1\right)\mathrm{B} - \frac{h^2 + 1}{\mathrm{A}^2} + 1 = 0 \qquad (4)$$

or as in § 20 with

$$\frac{h^2 - 1}{\mathrm{A}^2} = 1 + \frac{4x}{(y+1)^2}, \quad \frac{h^2 + 1}{\mathrm{A}^2} = 1 + \frac{4xy}{(y+1)^3} \qquad (5),$$

$$\mathrm{B}^3 - \mathrm{B}^2 + \frac{4x}{(y+1)^2}\mathrm{B} - \frac{4xy}{(y+1)^3} = 0 \qquad (6);$$

so that $\mathrm{B} = \frac{2c}{y+1}$ reduces this to

$$2c^3 - (y+1)c^2 + 2xc - x = 0 \qquad (7),$$

as in (3), § 8; and with

$$x = -\frac{m^3\alpha}{\alpha - m}, \quad y = -\frac{(1 - 2m)\alpha}{\alpha - m}, \quad y + 1 = \frac{(2\alpha - 1)m}{\alpha - m} \qquad (8),$$

$$c = \frac{m\alpha}{\alpha - m}, \quad \mathrm{B} = \frac{2\alpha}{2\alpha - 1}, \quad \lambda = \frac{2\alpha + 1}{2\alpha - 1} \qquad (9).$$

We find that CLEBSCH'S k (CRELLE, 57, p. 105; L.M.S., 27, pp. 146, 185) is given by

$$k^2 = 4\alpha - 1 \qquad (10),$$

so that

$$\lambda = \frac{k^2 + 3}{k^2 - 1} \qquad (11),$$

and

$$\mathrm{A}^2 = \frac{(k^2 - 1)^2 (k^2 - h^2)}{4k^2} \qquad (12),$$

so that

$$\mu^2 = 4\frac{(k^2 - h)^2}{(k^2 - 1)^2(k^2 - h^2)}, \quad \nu^2 = 4\frac{(k^2 + h)^2}{(k^2 - 1)^2(k^2 - h^2)} \qquad (13);$$

and in quadratic factors

$$(k^2 - 1)\mathrm{T}_1 = -\left\{k + 1 - 2\sqrt{\frac{k+h}{k-h}}\,t + (k-1)t^2\right\}\left\{k - 1 - 2\sqrt{\frac{k-h}{k+h}}\,t + (k+1)t^2\right\} \qquad (14),$$

$$(k^2 - 1)\mathrm{T}_2 = \left\{k + 1 + 2\sqrt{\frac{k+h}{k-h}}\,t + (k-1)t^2\right\}\left\{k - 1 + 2\sqrt{\frac{k-h}{k+h}}\,t + (k+1)t^2\right\} \qquad (15).$$

Thus, as shown in the 'Bulletin de la Société Mathématique de France,' 29, the algebraical spherical catenary for $\mu = 10$, drawn stereoscopically by the late Mr. T. I. DEWAR, can be represented, by taking $k = \sqrt{\frac{5}{3}}$, $h = -\frac{1}{2}\sqrt{\frac{17}{3}}$, in the symmetrical form

$$\mathrm{N}\,(\tan\tfrac{1}{2}\theta e^{\psi i})^{3} = (\mathrm{B} + \mathrm{B}_1 t + \mathrm{B}_2 t^2 + \mathrm{B}_3 t^3)\,\sqrt{}\mathrm{T}_1$$
$$+ i\,(\mathrm{B} - \mathrm{B}_1 t + \mathrm{B}_2 t^2 - \mathrm{B}_3 t^3)\,\sqrt{}\mathrm{T}_2 \qquad (16),$$

$$\mathrm{B},\ \mathrm{B}_3 = \pm\sqrt{}51 + 7, \quad \mathrm{B}_1,\ \mathrm{B}_2 = \pm 3\sqrt{}51 + 19, \quad \mathrm{N} = 192\sqrt{}3 \qquad (17).$$

27. These expressions for $\mathrm{I}(v)$ are applicable immediately to the construction of a series of quasi-algebraical Poinsot herpolhodes, in continuation of the one invented by HALPHEN (' F.E.,' 2, p. 279).

Making a digression on the motion of a rigid body about a fixed point under no forces, as illustrated by the motion of a body about its centre of gravity when tossed in the air, POINSOT'S polhode and herpolhode are obtained by rolling the momental ellipsoid on the invariable plane.

The equation of the momental ellipsoid can be written

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 = \mathrm{D}\delta^2 \qquad (1),$$

where δ denotes the distance of the invariable plane from the centre of the momental ellipsoid; and then the polhode will be the intersection with the coaxial quadric

$$\mathrm{A}^2x^2 + \mathrm{B}^2y^2 + \mathrm{C}^2z^2 = \mathrm{D}^2\delta^2 \qquad (2),$$

and the direction cosines of the invariable line will be

$$\frac{\mathrm{A}x}{\mathrm{D}\delta},\quad \frac{\mathrm{B}y}{\mathrm{D}\delta},\quad \frac{\mathrm{C}z}{\mathrm{D}\delta} \qquad (3).$$

Denoting by ρ, ϖ the polar co-ordinates for the herpolhode in the invariable plane,

$$x^2 + y^2 + z^2 = \rho^2 + \delta^2 \qquad (4),$$

and by solution of these equations (1), (2), (4),

$$\left(1 - \frac{\mathrm{A}}{\mathrm{B}}\right)\left(1 - \frac{\mathrm{A}}{\mathrm{C}}\right)x^2 = \rho^2 + \left(1 - \frac{\mathrm{D}}{\mathrm{B}}\right)\left(1 - \frac{\mathrm{D}}{\mathrm{C}}\right)\delta^2$$
$$= \rho^2 - \rho_a^2 \text{ suppose, \&c.} \qquad (5),$$

and then

$$\left\{\frac{(\mathrm{B}-\mathrm{C})\,(\mathrm{C}-\mathrm{A})\,(\mathrm{A}-\mathrm{B})}{\mathrm{ABC}}\,xyz\right\}^2 = \tfrac{1}{4}\,\mathrm{R} \qquad (6),$$

where

$$\mathrm{R} = 4\,(\rho_a^2 - \rho^2)\,(\rho_b^2 - \rho^2)\,(\rho_c^2 - \rho^2) \qquad (7).$$

Denoting the component angular velocities about the principal axes by p, q, r, and about the invariable line by h,

$$\frac{p}{x} = \frac{q}{y} = \frac{r}{z} = \frac{h}{\delta} = \frac{n}{k} \qquad (8),$$

and DARBOUX's equations (DESPEYROUS, 'Cours de mécanique,' II., note 17).

$$\frac{p^2}{a} + \frac{q^2}{b} + \frac{r^2}{c} = h, \quad \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 1 \tag{9}$$

are identified with our notation by putting

$$\mathrm{A}a = \mathrm{B}b = \mathrm{C}c = \mathrm{D}h \tag{10}$$

and

$$\mathrm{A}p^2 + \mathrm{B}q^2 + \mathrm{C}r^2 = \mathrm{D}h^2 \tag{11}$$

$$\mathrm{A}^2p^2 + \mathrm{B}^2q^2 + \mathrm{C}^2r^2 = \mathrm{D}^2h^2 \tag{12}$$

are two integrals of EULER's equations

$$\mathrm{A}\frac{dp}{dt} = (\mathrm{B} - \mathrm{C})\,qr, \ldots \tag{13}$$

or, as they may be written,

$$\mathrm{A}\frac{dx}{dt} = n\,(\mathrm{B} - \mathrm{C})\,\frac{yz}{k}, \ldots \tag{14}$$

Then

$$\begin{aligned} \frac{d\rho^2}{dt} &= 2\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) \\ &= 2n\left(\frac{\mathrm{B}-\mathrm{C}}{\mathrm{A}} + \frac{\mathrm{C}-\mathrm{A}}{\mathrm{B}} + \frac{\mathrm{A}-\mathrm{B}}{\mathrm{C}}\right)\frac{xyz}{k} \\ &= -\,2n\,\frac{(\mathrm{B}-\mathrm{C})(\mathrm{C}-\mathrm{A})(\mathrm{A}-\mathrm{B})}{\mathrm{ABC}}\,\frac{xyz}{k} = \frac{n}{k}\sqrt{\mathrm{R}} \end{aligned} \tag{15}$$

$$nt = \int_{\rho_3}^{\rho} \frac{k\,d\rho^2}{\sqrt{\mathrm{R}}} \tag{16}$$

supposing

$$\rho_3^2 > \rho^2 > \rho_2^2 > 0 > \rho_1^2 \tag{17}$$

and inverting the integral,

$$\rho^2 = \rho_2^2\,\mathrm{sn}^2\,(\mathrm{K} - mt) + \rho_3^2\,\mathrm{cn}^2\,(\mathrm{K} - mt), \quad \frac{m}{n} = \sqrt{\frac{\rho_3^2 - \rho_1^2}{k^2}} \tag{18}$$

Again, projecting on the invariable plane the areas swept out on the co-ordinate planes

$$\rho^2\frac{d\varpi}{dt} = \frac{\mathrm{A}x}{\mathrm{D}\delta}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + \ldots + \ldots \tag{19}$$

and

$$\begin{aligned} y\frac{dz}{dt} - z\frac{dy}{dt} &= \frac{n}{k}\left(\frac{\mathrm{A}-\mathrm{B}}{\mathrm{C}}\,xy^2 + \frac{\mathrm{A}-\mathrm{C}}{\mathrm{B}}\,xz^2\right) \\ &= \frac{n\delta^2}{k}\,\frac{\mathrm{D}\,(\mathrm{A}-\mathrm{D})}{\mathrm{BC}}\,x \end{aligned} \tag{20}$$

so that

$$\rho^2 \frac{d\varpi}{dt} = \frac{n\delta}{k} \left(\frac{A - D}{BC} Ax^2 + \frac{B - D}{CA} By^2 + \frac{C - D}{AB} Cz^2 \right)$$

$$= \frac{n\,\delta}{k} \frac{A^3x^2 + B^3y^2 + C^3z^2 - D^3\,\delta^2}{ABC} \qquad (21),$$

and this reduces to

$$\rho^2 \frac{d\varpi}{dt} = \frac{n\,\delta}{k} \left\{ \rho^2 + \frac{(A - D)\,(B - D)\,(C - D)}{ABC} \delta^2 \right\} \qquad (22);$$

so that

$$\varpi = \frac{\delta}{k} nt + \frac{(A - D)\,(B - D)\,(C - D)}{ABC} \delta^3 \int_{\rho_2}^{\rho} \frac{d\rho^2}{\rho^2 \sqrt{R}} \qquad (23).$$

Putting

$$\frac{\rho^2}{k^2} = \frac{\sigma - s}{M^2}, \quad \frac{\rho_a{}^2 - \rho^2}{k^2} = \frac{s - s_a}{M^2}, \quad \frac{R}{k^6} = \frac{S}{M^6} \qquad (24),$$

and $\rho = 0$, with $s = \sigma$,

$$R_0 = 4\rho_a{}^2\rho_b{}^2\rho_c{}^2 = -4 \left\{ \frac{(A - D)\,(B - D)\,(C - D)}{ABC} \right\}^2 \delta^6 = \frac{k^6\Sigma}{M^6} \qquad (25);$$

so that

$$\frac{(A - D)\,(B - D)\,(C - D)}{ABC} \delta^3 = -\frac{\frac{1}{2} k^3 \sqrt{-\Sigma}}{M^3} \qquad (26),$$

and from (6) § 1

$$\varpi = \frac{\delta}{k} nt - \int_{s}^{s_2} \frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma - s} \frac{ds}{\sqrt{S}}$$

$$= \left(\frac{\delta}{k} - \frac{Pv}{M} \right) nt - I\,(v) \qquad (27),$$

or

$$pt - \varpi = I\,(v) \qquad (28),$$

where

$$\frac{p}{n} = \frac{\delta}{k} - \frac{Pv}{M} \qquad (29);$$

also

$$nt = \int_{s}^{s_2} \frac{M\,ds}{\sqrt{S}} \qquad (30).$$

Then

$$e^{2iI(v)} = \frac{\theta\,(u - v)}{\theta\,(u + v)} \qquad (31),$$

and

$$M^2 \frac{\rho^2}{k^2} = \sigma - s = \wp v - \wp u$$

$$= \frac{\sigma\,(u + v)\,\sigma\,(u - v)}{\sigma^2 u \sigma^2 v} = \frac{\theta^2 0 \theta\,(u + v)\,\theta\,(u - v)}{\lambda^2 \theta^2 u \theta^2 v} \qquad (32),$$

where λ is a homogeneity factor, $\lambda = m/n$; so that as a quasi-Lamé function

$$M \frac{\rho}{k} e^{(pt - \varpi)i} = \frac{\theta 0 \theta\,(u - v)}{\lambda \theta u \theta v}$$

$$= \{(s^n + P_1 s^{n-1} + \ldots + P_n) \sqrt{(c^2 - s)}$$

$$+ i\,(Q_1 s^{n-1} + \ldots + Q_n) \sqrt{(a^2 - s \,.\, s - b^2)}\}^{\frac{1}{2n+1}} \qquad (33)$$

in the pseudo-elliptic case; and cancelling the secular term pt by making $p = 0$ gives a purely algebraical herpolhode.

With the degree halved by the use of the variable y, as in (8) § 24,

$$\mathrm{M}\frac{\rho}{k}e^{(pt-\varpi)i} = \frac{[(y^{n-1} + \mathrm{A}_1 y^{n-2} + \ldots)\sqrt{\mathrm{Y}_1} + i(y^{n-1} - \mathrm{A}_1 y^{n-2} + \ldots)\sqrt{\mathrm{Y}_2}]^{\frac{2}{2n+1}}}{\sqrt{(y^2 - 1)}} \quad (34),$$

$$\mathrm{M}^2\frac{\rho^2}{k^2} = \frac{y^2}{y^2 - 1} \quad (35).$$

In the associated motion of a top, the projection on a horizontal plane of a point on the axis describes the hodograph of a herpolhode traced out by the axis of resultant angular momentum of the top; and thus EULER's angles θ and ψ for the top are connected by a relation of the form

$$\tfrac{1}{2}\mathrm{M}^2 \sin\theta e^{\psi i} = -i\frac{d}{dt}(\rho e^{\varpi i}) \quad (36)$$

('Science,' Dec. 20, 1901; 'Annals of Mathematics,' vol. 5, Series II., 1903).

We may dispense with POINSOT's rolling surface and consider the polhode as a material line, or as the edge of a material cone, as in Dr. FR. SCHILLING's model, constructed by MARTIN SCHILLING, Halle a. S., and this is rolled on a fixed plane.

According to M. DE LA GOURNERIE, the polhode is also a line of curvature, the intersection of two confocal surfaces, an ellipsoid and hyperboloid of two sheets; and θ will now be the angle between the generating lines of the hyperboloid of one sheet, one generator being perpendicular to the fixed plane, so that the other generator moves parallel to the axis of the top in the associated motion (DARBOUX in DESPEYROUS, 'Mécanique,' note 18).

28. To utilise for $\mu = 4n + 2$ the preceding results for odd order $2n + 1$ of μ, where

$$t = \frac{ma}{a - m}y \quad (1),$$

$$\mathrm{T}_1, \mathrm{T}_2 = \frac{m^3 a^3}{(a - m)^3}(y \pm 1)\left[2y^2 \mp \frac{y}{a} + \frac{(1 - 2m)}{a}\right] \quad (2),$$

put

$$s(\omega) - s(4n + 2)v = \infty, \quad \mathrm{D}_{2n+1} = 0 \quad (3),$$

and thence express a, m, ... in terms of a new parameter, and thereby express the integral I $(4v)$ of $\mu = 4n + 2$.

Returning then to the v of $\mu = 2n + 1$, which, when normalised to LEGENDRE's form, can be written

$$v = f\mathrm{K}'i, \quad f = \frac{2m}{2n + 1} \quad (4),$$

and replacing $\gamma_\mu = 0$ by its equivalent $\mathrm{D}_{2n+1} = 0$, the series of functions

$$a_1 = 1 + \frac{1}{\alpha - m} \tag{5},$$

$$a_2 = 1 - \frac{m-1}{\alpha} \tag{6},$$

$$a_3 = \left(1 + \frac{1}{\alpha - m}\right)\frac{m^2}{(1-m)^2} \tag{7},$$

$$a_4 = \left(1 - \frac{m-1}{\alpha}\right)\left\{\frac{(1-2m)\,\alpha}{(1-2m)\,\alpha - m\,(1-m)}\right\}^2 \tag{8},$$

. . . .

$$a_{2n} = \left(1 - \frac{m-1}{\alpha}\right)\left(\frac{\mathrm{D}_{2n}}{\mathrm{N}_{2n}}\right)^2 \tag{9},$$

$$a_{2n+1} = \left(1 + \frac{1}{\alpha - m}\right)\left(\frac{\mathrm{D}_{2n+1}}{\mathrm{N}_{2n+1}}\right)^2 \tag{10},$$

is such that

$$a_r = \kappa^2 \mathrm{tn}^4 rf\mathrm{K}' = \left[\frac{\mathrm{H}\,(rf\mathrm{K}')}{\mathrm{H}\,(1-rf)\,\mathrm{K}'}\right]^4, \text{ or } -\frac{\kappa'^2}{\kappa^2}\mathrm{cn}^4(1-rf)\,\mathrm{K}' = -\left[\frac{\mathrm{H}\,(rf\mathrm{K}')}{\Theta\,(1-rf)\,\mathrm{K}'}\right]^4 \tag{11},$$

so that Jacobi's division-values of the Second Stage can be calculated.

The Weierstrassian Sigma Functions of the First Stage in § 4 can now be converted into Jacobian Theta Functions of the Second Stage by the relations on p. 52 of 'Formeln und Lehrsätze'; and forming the series of functions

$$c_1 = x^{-\frac{1}{3}}\lambda^{-\frac{1}{2n+1}}\, m\sqrt{\frac{\alpha}{\alpha - m}} \tag{12},$$

$$c_2 = x^{-\frac{1}{3}}\lambda^{-\frac{4}{2n+1}}\,\frac{m\alpha}{\alpha - m} \tag{13},$$

$$c_3 = x^{-\frac{1}{3}}\lambda^{-\frac{9}{2n+1}}\,(1-m)\sqrt{\frac{\alpha}{\alpha - m}}\,\gamma_3 \tag{14},$$

. . . .

$$c_{2r} = x^{-\frac{1}{3}}\lambda^{-\frac{4r^2}{2n+1}}\,\frac{m\alpha}{\alpha - m}\,\frac{\mathrm{N}_{2r}}{\mathrm{D}_{2r}}\,\gamma_{2r} \tag{15},$$

$$c_{2r+1} = x^{-\frac{1}{3}}\lambda^{-\frac{(2r+1)^2}{2n+1}}\, m\sqrt{\frac{\alpha}{\alpha - m}}\cdot\frac{\mathrm{N}_{2r+1}}{\mathrm{D}_{2r+1}}\,\gamma_{2r+1} \tag{16},$$

we find that

$$c_r = \frac{\Theta\, rf\mathrm{K}'}{\Theta 0} \text{ or } \frac{\mathrm{H}rf\mathrm{K}'}{\mathrm{H}\mathrm{K}'} \tag{17},$$

or one of its linear transformations; and $\sqrt{\dfrac{\alpha}{\alpha - m}}$ is always rational in α and m, in consequence of the relation

$$s\,(\omega) - s\,(nv) = s\,(\omega) - s\,(n+1)\,v \tag{18}.$$

The function c is the one that should be tabulated numerically, as

$$\operatorname{sn} u = \frac{\mathrm{H}u}{\mathrm{HK}} \Big/ \frac{\Theta u}{\Theta \mathrm{K}}, \quad \operatorname{cn} u = \frac{\mathrm{H}(\mathrm{K}-u)}{\mathrm{HK}} \Big/ \frac{\Theta u}{\Theta 0}, \quad \operatorname{dn} u = \frac{\Theta(\mathrm{K}-u)}{\Theta \mathrm{K}} \Big/ \frac{\Theta u}{\Theta 0} \qquad (19),$$

Also

$$\frac{\mathrm{P}(v)}{\sqrt{(a^2-b^2)}} = \operatorname{zs} f\mathrm{K}' \qquad (20),$$

or one of its transformations, depending on the region.

29. $\mu = 6, \quad f = \frac{1}{3}$ or $\frac{2}{3}$.

Derive from $\mu = 3$ in (3) § 6, by putting $c = a^2(2a-1)$, so that

$$\mathrm{I}(v) = \frac{2}{3}\cos^{-1}\frac{\sqrt{(t+a)}\sqrt{\{2t^2-(2a-1)t+2a^2-a\}}}{2t^{\frac{3}{2}}}$$

$$= \frac{2}{3}\sin^{-1}\frac{\sqrt{(t-a)}\sqrt{\{2t^2+(2a-1)t+2a^2-a\}}}{2t^{\frac{3}{2}}}$$

$$= \int_a^v \frac{\frac{1}{3}t^2+2a^3-a^2}{t} \cdot \frac{dt}{\sqrt{\{4t^6-(t^2+2a^3-a^2)^2\}}} \qquad (1).$$

Next put

$$a = \frac{1}{2}\frac{b^2-1}{b^2+3} \qquad (2),$$

$$t_1 = \frac{b+1}{b^2+3}, \quad t_2 = \frac{1}{2}\frac{b^2-1}{b^2+3}, \quad t_3 = \frac{b-1}{b^2+3} \qquad (3),$$

$$\kappa^2 = \frac{(b-1)^3(b+3)}{16b}, \quad \kappa'^2 = \frac{(b+1)^3(-b+3)}{16b} \qquad (4),$$

$$\mathrm{L}(2)^{24} = \frac{4096b^2}{(b^2-1)^3(b^2-9)} \qquad (5).$$

Then from (8), § 23, in the region $3 > b > 1$,

$$\operatorname{cn}\tfrac{2}{3}\mathrm{K}' = \frac{b-1}{b+1}, \quad \operatorname{dn}\tfrac{2}{3}\mathrm{K}' = \frac{b-1}{2}, \quad \operatorname{sn}\tfrac{1}{3}\mathrm{K}' = \frac{2}{b+1} \qquad (6),$$

so that there results the well-known relation

$$\operatorname{cn}\tfrac{2}{3}\mathrm{K}' + \operatorname{sn}\tfrac{1}{3}\mathrm{K}' = 1 \qquad (7),$$

and in LEGENDRE, 1, chap. vi., p. 27,

$$x = \sin\phi = \operatorname{sn}\tfrac{1}{3}\mathrm{K} = \frac{2}{b+1}, \quad \kappa^2 = \frac{1-2x}{-2x^3+x^4} \qquad (8).$$

The following table shows the six linear transformations of the division-values for $\mu = 6$, obtained in accordance with the preceding theory by the substitutions

$$\left(b, \quad \frac{b+3}{b-1}, \quad -b\right) \qquad (9);$$

the accents are omitted from K by changing to the complementary modulus.

The numerical value $b = \sqrt{3}$ gives a modular angle 75° ; while $b = \sqrt{5}$ corresponds to the functions required for $\mu = 10$.

Region.	A. $\infty > b > 3.$	B. $3 > b > 1.$	C. $1 > b > 0.$	D. $0 > b > -1.$	E. $-1 > b > -3.$	F. $-3 > b > -\infty$
$\frac{(b+1)^3(-b+3)}{16b}$	$-\frac{\kappa'^2}{\kappa^2}$	κ^2	$-\frac{\kappa^2}{\kappa'^2}$	$\frac{1}{\kappa'^2}$	κ'^2	$-\frac{\kappa'^2}{\kappa^2}$
$\frac{(b-1)^3(b+3)}{16b}$	$\frac{1}{\kappa^2}$	κ'^2	$\frac{1}{\kappa'^2}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	$\frac{1}{\kappa^2}$
$\frac{2}{b+1}$	$\mathrm{cn}\,\frac{2}{3}\mathrm{K}$	$\mathrm{sn}\,\frac{1}{3}\mathrm{K}$	$\mathrm{ns}\,\frac{1}{3}\mathrm{K}$	$\mathrm{nc}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{nd}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{dn}\,\frac{2}{3}\mathrm{K}$
$\frac{b-1}{b+1}$	$\mathrm{sn}\,\frac{1}{3}\mathrm{K}$	$\mathrm{cn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{dn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{nd}\,\frac{2}{3}\mathrm{K}$	$\mathrm{nc}\,\frac{2}{3}\mathrm{K}$	$\mathrm{ns}\,\frac{1}{3}\mathrm{K}$
$\frac{b-1}{2}$	$\mathrm{nd}\,\frac{2}{3}\mathrm{K}$	$\mathrm{dn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{cn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{sn}\,\frac{1}{3}\mathrm{K}$	$-\mathrm{ns}\,\frac{1}{3}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{3}\mathrm{K}$
$\frac{-b^2+9}{12\sqrt{b}}$	$-\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$\frac{1}{\kappa}\,\mathrm{zc}\,\frac{2}{3}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zc}\,\frac{2}{3}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$-\frac{i}{\kappa'}\,\mathrm{zn}\,\frac{2}{3}\mathrm{K}$
$\frac{(b+3)\sqrt{b}}{6}$	$\frac{1}{\kappa'}\,\mathrm{zs}\,\frac{1}{3}\mathrm{K}$	$\mathrm{zs}\,\frac{1}{3}\mathrm{K}$	$\frac{1}{\kappa}\,\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\frac{i}{\kappa'}\,\mathrm{zn}\,\frac{1}{3}\mathrm{K}$
$\frac{(-6+3)\sqrt{b})}{6}$	$-\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\frac{1}{\kappa}\,\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$i\,\mathrm{zs}\,\frac{1}{3}\mathrm{K}$	$\frac{i}{\kappa'}\,\mathrm{zs}\,\frac{1}{3}\mathrm{K}$
$\frac{(b^2-1)^{\frac{1}{3}}}{2}$	$\frac{\Theta\,\frac{2}{3}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\mathrm{H}\,\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$-\frac{\mathrm{H}\,\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{2}{3}\mathrm{K}}{\Theta 0}$
$\frac{(b+1)^{\frac{1}{3}}}{(b-1)^{\frac{1}{3}}}$	$\frac{\Theta\,\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{2}{3}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{2}{3}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\mathrm{H}\,\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$-\frac{\mathrm{H}\,\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$
$\frac{(b-1)^{\frac{1}{3}}}{(b+1)^{\frac{1}{3}}}$	$\frac{\mathrm{H}\,\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$-\frac{\Theta\,\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\Theta\,\frac{2}{3}\mathrm{K}}{\Theta 0}$	$-\frac{\Theta\,\frac{2}{3}\mathrm{K}}{\Theta 0}$	$-\frac{\Theta\,\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$
Test by b	$2\sqrt{3}+3$	$\sqrt{3}$	$2\sqrt{3}-3$	$-2\sqrt{3}+3$	$-\sqrt{3}$	$-2\sqrt{3}-3$
or b	$\sqrt{5}+2$	$\sqrt{5}$	$\sqrt{5}-2$	$-\sqrt{5}+2$	$-\sqrt{5}$	$-\sqrt{5}-2$

30. $\mu = 10$, $f = \frac{1,\,2,\,3,\,4}{5}$.

$$D_5 = 0, \quad \alpha = \frac{m(1-m)^2}{1-2m}, \quad \alpha - m = \frac{m^3}{1-2m} \tag{1},$$

$$x = y = -\frac{(1-m)^2(1-2m)}{m^2} \tag{2}.$$

To identify with the results in L.M.S., 27, put

$$2m - 1 = \frac{1}{c}, \quad m = \frac{c+1}{2c}, \quad 1 - m = \frac{c-1}{2c} \tag{3};$$

then

$$T_1, T_2 = \left[t \pm \frac{(c-1)^2}{2c(c+1)}\right]\left[2t^2 \pm \frac{4c}{(c+1)^2}t + \frac{2(c-1)^2}{c(c+1)^3}\right] \tag{4},$$

and from (9), § 7,

$$\begin{aligned} I(4v) &= \frac{2}{5}\cos^{-1}\frac{\left\{t - \frac{(c-1)^2}{c(c+1)^2}\right\}\sqrt{T_1}}{2t^{\frac{5}{2}}} \\ &= \frac{2}{5}\sin^{-1}\frac{\left\{t + \frac{(c-1)^2}{c(c+1)^2}\right\}\sqrt{T_2}}{2t^{\frac{5}{2}}} \\ &= \int \frac{\frac{c^3 - c^2 + 7c - 3}{10c(c+1)^2}t^2 - \frac{(c-1)^4}{2c^2(c+1)^4}}{t^2}\frac{dt^2}{\sqrt{T_1T_2}} \end{aligned} \tag{5}.$$

Put

$$t = \frac{(c-1)^2}{2c(c+1)}y, \quad \text{and} \quad y^2 = \frac{s(v) - s}{s(\omega) - s} \tag{6},$$

with

$$s(v) = 8c(c+1)^2(c-1), \quad s(\omega) = (c+1)^2(c-1)^4 \tag{7},$$

and we obtain $I(v)$ in the form employed in L.M.S., 27, p. 601, derived by putting $N_5 = 0$, with

$$P(v) = \tfrac{1}{5}(c+3)(-c^2+4c+1), \quad Q(v) = 4c(c+1)^2(c-1)(-c^2+4c+1) \tag{8};$$

and in Region II., $f = \frac{1}{5}$ or $\frac{3}{5}$,

$$s_1 = 4(c^2 + \sqrt{C})^2, \quad s_2 = (c+1)^2(c-1)^4, \quad s_3 = 4(c^2 - \sqrt{C})^2 \tag{9},$$

where

$$C = c^3 + c^2 - c \tag{10},$$

and

$$L(2)^{24} = \frac{4096c^4C}{(c^2-1)^5(c^2-4c-1)} \tag{11}.$$

This c is connected with FRICKE's τ and t (' Diss.,' Leipzig, 1886) by the relation

$$c = \frac{\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^2}{\tau - \frac{\sqrt{5}+1}{2\sqrt{5}}}, \quad t = \frac{-c^2+4c+1}{5c} \tag{12}$$

The following table, analogous to the preceding one for $\mu = 6$, gives the division-values for $\mu = 10$, the six transformations being derived from the substitution

$$\left\{c, \quad \left[\frac{\sqrt{(c^2+c-1)} \pm \sqrt{c}}{c-1}\right]^2, \quad -\frac{1}{c}\right\} \tag{13};$$

and it may be compared with the corresponding results given by F. MÜLLER, ' Archiv der Mathematik und Physik,' p. 161, 1884, ROHDE, ' Archiv,' 1886, p. 138, and by W. GÖRING, ' Math. Ann.,' 7, p. 311, 1874, working on GAUSS's unpublished memoirs.

For the application of this integral to a case of motion of the top, consult the ' Annals of Mathematics,' vol. 5, 1903.

The numerical value $c = 2$ will serve for verification; this corresponds to the case of ten cusps at intervals of $\frac{1}{5}\pi$ in the associated motion of the top (L.M.S., 27, p. 602); and $c = 2(\sin 36^\circ \pm \sin 18^\circ)$ will make $K'/K = \sqrt{5}$ or 1, cases of Complex Multiplication.

Region.	A. $\infty > c > \sqrt{5}+2.$	B. $\sqrt{5}+2 > c > 1.$
$\frac{(2c-1+\sqrt{C})(2c+1-\sqrt{C})(\sqrt{C}+1)^2}{16c^2\sqrt{C}}$	$\frac{1}{\kappa^2}$	κ^2
$\frac{(2c+1+\sqrt{C})(-2c+1+\sqrt{C})(\sqrt{C}-1)^2}{16c^2\sqrt{C}}$	$-\frac{\kappa'^2}{\kappa^2}$	κ'^2
$\frac{2c}{c^2+\sqrt{C}}$	$\text{cn}\,\frac{4}{5}\text{K}$	$\text{sn}\,\frac{1}{5}\text{K}$
$\frac{2c}{\sqrt{C}+1}$	$\text{cn}\,\frac{2}{5}\text{K}$	$\text{sn}\,\frac{3}{5}\text{K}$
$\frac{\sqrt{C}-1}{\sqrt{C}+1}$	$\text{sn}\,\frac{3}{5}\text{K}$	$\text{cn}\,\frac{2}{5}\text{K}$
$\frac{\sqrt{C}-1}{2c}$	$\text{nd}\,\frac{2}{5}\text{K}$	$\text{dn}\,\frac{2}{5}\text{K}$
$\frac{c^2-\sqrt{C}}{c^2+\sqrt{C}}$	$\text{sn}\,\frac{1}{5}\text{K}$	$\text{cn}\,\frac{4}{5}\text{K}$
$\frac{c^2-\sqrt{C}}{2c}$	$\text{nd}\,\frac{4}{5}\text{K}$	$\text{dn}\,\frac{4}{5}\text{K}$
$\frac{(c+3)(-c^2+4c+1)}{20c\sqrt[4]{C}}$	$-\frac{1}{\kappa'}\text{zn}\,\frac{4}{5}\text{K}$	$\text{zn}\,\frac{1}{5}\text{K}$
$\frac{(3c-1)(-c^2+4c+1)}{20c\sqrt[4]{C}}$	$-\frac{1}{\kappa'}\text{zn}\,\frac{2}{5}\text{K}$	$\text{zn}\,\frac{3}{5}\text{K}$
$\frac{c^3-c^2+7c-3}{20c\sqrt[4]{C}}$	$\frac{1}{\kappa'}\text{zs}\,\frac{4}{5}\text{K}$	$\text{zs}\,\frac{4}{5}\text{K}$
$\frac{3c^3+7c^2+c+1}{20c\sqrt[4]{C}}$	$\frac{1}{\kappa'}\text{zs}\,\frac{2}{5}\text{K}$	$\text{zs}\,\frac{2}{5}\text{K}$
$\frac{(c+1)^{\frac{1}{5}}(c-1)^{\frac{4}{5}}}{2c^{\frac{3}{5}}}$	$\frac{\Theta\,\frac{4}{5}\text{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{5}\text{K}}{\Theta\text{K}}$
$\frac{(c+1)^{\frac{4}{5}}(c-1)^{\frac{1}{5}}}{2c^{\frac{3}{5}}}$	$\frac{\Theta\,\frac{2}{5}\text{K}}{\Theta 0}$	$\frac{\Theta\,\frac{3}{5}\text{K}}{\Theta\text{K}}$

C. $1 > c > \frac{\sqrt{5}-1}{2}$.	D. $0 > c > -\sqrt{5}+2$.	E. $-\sqrt{5}+2 > c > -1$.	F. $-1 > c > -\frac{\sqrt{5}+1}{2}$.
$-\frac{\kappa^2}{\kappa'^2}$	$\frac{1}{\kappa'^2}$	κ'^2	$-\frac{\kappa'^2}{\kappa^2}$
$\frac{1}{\kappa'^2}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	$\frac{1}{\kappa^2}$
$\mathrm{ns}\,\frac{1}{5}\mathrm{K}$	$\mathrm{dn}\,\frac{2}{5}\mathrm{K}$	$\mathrm{nd}\,\frac{2}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{5}\mathrm{K}$
$\mathrm{ns}\,\frac{3}{5}\mathrm{K}$	$-\mathrm{dn}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nd}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{4}{5}\mathrm{K}$
$-\mathrm{dn}\,\frac{2}{5}\mathrm{K}$	$-\mathrm{ns}\,\frac{1}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nd}\,\frac{4}{5}\mathrm{K}$
$-\mathrm{cn}\,\frac{2}{5}\mathrm{K}$	$\mathrm{nc}\,\frac{4}{5}\mathrm{K}$	$\mathrm{ns}\,\frac{1}{5}\mathrm{K}$	$\mathrm{sn}\,\frac{1}{5}\mathrm{K}$
$\mathrm{dn}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{ns}\,\frac{3}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{5}\mathrm{K}$	$\mathrm{nd}\,\frac{2}{5}\mathrm{K}$
$\mathrm{cn}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{5}\mathrm{K}$	$-\mathrm{ns}\,\frac{3}{5}\mathrm{K}$	$-\mathrm{sn}\,\frac{3}{5}\mathrm{K}$
$-\frac{1}{\kappa}\mathrm{zs}\,\frac{1}{5}\mathrm{K}$	$\frac{i}{\kappa'}\mathrm{zn}\,\frac{2}{5}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{3}{5}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zc}\,\frac{2}{5}\mathrm{K}$
$-\frac{1}{\kappa}\mathrm{zs}\,\frac{3}{5}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zn}\,\frac{4}{5}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{1}{5}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zc}\,\frac{4}{5}\mathrm{K}$
$\frac{1}{\kappa}\mathrm{zs}\,\frac{4}{5}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zs}\,\frac{2}{5}\mathrm{K}$	$i\,\mathrm{zs}\,\frac{2}{5}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zs}\,\frac{2}{5}\mathrm{K}$
$\frac{1}{\kappa}\mathrm{zs}\,\frac{2}{5}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zs}\,\frac{4}{5}\mathrm{K}$	$-i\,\mathrm{zs}\,\frac{4}{5}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zs}\,\frac{4}{5}\mathrm{K}$
$\frac{\mathrm{H}\,\frac{1}{5}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{2}{5}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{3}{5}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\mathrm{H}\,\frac{3}{5}\mathrm{K}}{\mathrm{HK}}$
$-\frac{\mathrm{H}\,\frac{3}{5}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{4}{5}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{5}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\mathrm{H}\,\frac{1}{5}\mathrm{K}}{\mathrm{HK}}$

31. $\mu = 7, 14$; $f = \frac{1, 2, 3, 4, 5, 6}{7}$;

$$D_7 = (1-2m)^3\alpha^2 - m(1-2m)(1-3m)\alpha - m^4(1-m)^2 = 0 \qquad (1);$$

and on putting

$$1 - 2m = p, \quad (1-2m)\alpha = m(1-m)\beta, \quad 2\beta = 1 + \frac{1}{q} \qquad (2),$$

the relation becomes

$$p^2q^2 + pq - p - q = 0 \qquad (3),$$

an elliptic-function addition equation, connecting

$$p = p(u) \quad \text{and} \quad q = p(u + \tfrac{2}{3}\omega) \qquad (4),$$

$$u = \int \frac{dp}{\sqrt{P}}, \quad P = 4p^3 + (p-1)^2 = (p+1)(4p^2 - 3p + 1) \qquad (5),$$

and

$$p(\tfrac{2}{3}\omega) = 0, \quad p(\tfrac{1}{3}\omega) = 1 \qquad (6).$$

Thence

$$\alpha = \frac{(p^2-1)(-3p+1+\sqrt{P})}{16p^2} \qquad (7),$$

$$\frac{\alpha}{\alpha - m} = \frac{(p+1)\{p\sqrt{(p+1)} - \sqrt{(4p^2-3p+1)}\}^2}{(p-1)^4} \qquad (8),$$

$$L(2)^{24} = \frac{p(p-1)\{7p(p^2-1) - (p^2+p+2)\sqrt{P}\}}{2(p^3 - 9p^2 - p + 1)} \qquad (9).$$

The connexion with KIEPERT's results in 'Math. Ann.,' 32, p. 87, has been given in L.M.S., 27, p. 438; here

$$\xi = 8\xi_4 = \frac{(p+1)(p-1)^2}{p^2} \qquad (10),$$

$$\frac{d\xi}{\sqrt{(4\xi^3 + 13\xi^2 + 32\xi)}} = \frac{dp}{\sqrt{P}} = du \qquad (11),$$

a transformation of the Third Order; and

$$\int_{-1}^{0} \frac{dp}{\sqrt{P}} = \int_0^1 = \int_1^{\infty} = \int_0^{\infty} \frac{d\xi}{\xi'} = \tfrac{1}{3}\omega \qquad (12),$$

suppose, where ω is the real half-period of the elliptic function $p = p(u)$ of the elliptic argument u, and now

$$\xi_1 = \xi(u - \tfrac{1}{9}\omega), \quad \frac{1}{\xi_4} = \xi(u - \tfrac{1}{3}\omega) \qquad (13),$$

$$\xi\frac{2r\omega}{9} = 1, \quad p^3 - 2p^2 - p + 1 = 0 \qquad (14).$$

the roots of which are

$$p = p\left(\frac{2r\omega}{9}\right) = \tfrac{1}{2}\sec\frac{1,\ 3,\ 5}{7}\pi \qquad (15),$$

$$\xi\frac{r\omega}{9} = 8, \quad p^3 - 9p^2 - p + 1 = 0 \qquad (16),$$

the roots of which are derived from the preceding, by the substitution

$$\left(p + 1, \ \frac{2}{p + 1}\right) \qquad (17).$$

Thence the contour of the period parallelogram is divided into segments of $\frac{1}{9}\omega$, and a similar table of division-values for $\mu = 14$ can be constructed for the corresponding regions of p and compared with the results of GÖRING and MÜLLER; this must be reserved for the present.

The associated τ parameters of GIERSTER ('Math. Ann.,' 14) have simple numerical values:

$$\tau_{18} = -1, \quad \tau_{12} = 2, \quad \tau_9 = \tfrac{4}{3}, \quad \tau_6 = -\tfrac{7}{2}, \quad \tau_4 = -2, \quad \tau_3 = \tfrac{28}{27}, \quad \tau_2 = \tfrac{7}{128} \quad (18).$$

The parameter z, employed in § 9 (2), is given in terms of p by the relation

$$z = \frac{y - x}{y} = \frac{s(3r) - s(2r)}{s(3r) - s(r)}$$

$$= \frac{(1 - m)^2 - \dfrac{m^2\alpha}{\alpha - m}}{1 - 2m} = \frac{(p + 1)(-3p + 1 + \sqrt{P})}{2(p - 1)^2} \qquad (19);$$

thence h_1, h_2 in § 9 (6) can be determined, and T_1, T_2 in factors, as required for the Second Stage of $\mu = 14$, in terms of p and $\sqrt{P}$.

Changing from the variable t to y, in § 28 (1), we find

$$I = \int^{\frac{2P(r)}{M}} \frac{y^2 - Q}{y} \frac{dy}{\sqrt{(Y_1 Y_2)}}$$

$$= \frac{2}{7}\cos^{-1}\frac{y^2 + A_1 y + A_2}{y^2}\sqrt{\tfrac{1}{2}Y_2}$$

$$= \frac{2}{7}\sin^{-1}\frac{y^2 - A_1 y + A_2}{y^2}\sqrt{\tfrac{1}{2}Y_1} \qquad (20).$$

$$Y_2 = (y - 1)(y^2 - C_1 y + C_2)$$

$$Y_1 = (y + 1)(y^2 + C_1 y + C_2) \qquad (21),$$

$$C_1 = -\frac{1}{2\alpha} = -\frac{2p(3p - 1 + \sqrt{P})}{(p + 1)(p - 1)^3}$$

$$C_2 = -pC_1 = \frac{2p^2(3p - 1 + \sqrt{P})}{(p + 1)(p - 1)^3} \qquad (22).$$

$$\mathrm{A}_1 = -\frac{\{\sqrt{(4p^2-3p+1)}-\sqrt{(p+1)}\}^2}{2(p^2-1)}$$

$$\mathrm{A}_2 = \frac{2p\mathrm{A}_1}{p-1} \qquad (23),$$

$$\frac{\mathrm{P}(r)}{\mathrm{M}} = \frac{-5p^4+12p^3-4p^2+2p-1+2(p^2-p+1)\sqrt{\mathrm{P}}}{14(p+1)(p-1)^3} \qquad (24),$$

$$\mathrm{Q} = \frac{1-2m}{2\alpha} = \frac{2p^2(3p-1+\sqrt{\mathrm{P}})}{(p+1)(p-1)^3} \qquad (25).$$

Test by $p = \sqrt{2}-1 = p\tfrac{1}{2}\omega, \quad \dfrac{\mathrm{K}'}{\mathrm{K}} = \sqrt{14}.$

For the elliptic-section values the region may be selected in which, in accordance with § 28,

$$\kappa^2\mathrm{tn}^4\tfrac{2}{7}\mathrm{K}' = \alpha_1 = \frac{(p+1)^{\frac{1}{2}}\{(p^3-7p^2+3p-1)\sqrt{(p+1)}+4p\sqrt{(4p^2-3p+1)}\}}{(p-1)^5} \qquad (26),$$

$$\kappa^2\mathrm{tn}^4\tfrac{5}{7}\mathrm{K}' = \frac{1}{\alpha_1} = \frac{(p-1)\{(p^3-7p^2+3p-1)\sqrt{(p+1)}-4p\sqrt{(4p^2-3p+1)}\}}{(p^3-9p^2-p+1)(p+1)^{\frac{1}{2}}} \qquad (27),$$

$$\kappa^2\mathrm{tn}^4\tfrac{4}{7}\mathrm{K}' = \alpha_2 = \frac{(p+1)^{\frac{1}{2}}\{(p^2+2p-1)\sqrt{(p+1)}+2p\sqrt{(4p^2-3p+1)}\}}{(p-1)^3} \qquad (28),$$

$$\kappa^2\mathrm{tn}^4\tfrac{3}{7}\mathrm{K}' = \frac{1}{\alpha_2} = \frac{(p-1)\{(p^2+2p-1)\sqrt{(p+1)}-2p\sqrt{(4p^2-3p+1)}\}}{(p^3-9p^2-p+1)(p+1)^{\frac{1}{2}}} \qquad (29),$$

$$\kappa^2\mathrm{tn}^4\tfrac{6}{7}\mathrm{K}' = \alpha_3 = \frac{(p^3-7p^2+3p-1)\sqrt{(p+1)}+4p\sqrt{(4p^2-3p+1)}}{(p-1)^3(p+1)^{\frac{1}{2}}} \qquad (30),$$

$$\kappa^2\mathrm{tn}^4\tfrac{1}{7}\mathrm{K}' = \frac{1}{\alpha_3} = \frac{(p+1)^{\frac{1}{2}}\{(p^3-7p^2+3p-1)\sqrt{(p+1)}-4p\sqrt{(4p^2-3p+1)}\}}{(p^3-9p^2-p+1)(p-1)} \qquad (31).$$

With

$$\gamma_7 = 0, \quad \lambda = \frac{\gamma_4}{\gamma_3} = \frac{y}{x^3}, \quad x^3\lambda^{\frac{1}{7}} = x^{\frac{4}{7}}y^{\frac{1}{7}} \qquad (32),$$

$$\left(\frac{\Theta\,\frac{2}{7}\mathrm{K}'}{\Theta 0}\right)^7 = c_1^{\,7} = \frac{m\alpha^2}{1-2m}\left(\frac{\alpha-m}{\alpha}\right)^{\frac{1}{2}} \qquad (33),$$

$$\left(\frac{\Theta\,\frac{4}{7}\mathrm{K}'}{\Theta 0}\right)^7 = c_2^{\,7} = \frac{m^4\alpha}{(1-2m)^7}\left(\frac{\alpha}{\alpha-m}\right) \qquad (34),$$

$$\left(\frac{\Theta\,\frac{6}{7}\mathrm{K}'}{\Theta 0}\right)^7 = c_3^{\,7} = \frac{m^9(1-m)^7}{(1-2m)^9\alpha^3}\left(\frac{\alpha}{\alpha-m}\right)^{\frac{1}{2}} \qquad (35).$$

32. $\mu = 18$, $f = \frac{1, 2, 4, 5, 7, 8}{9}$.

The relation $D_9 = 0$ becomes by the substitutions

$$\alpha = \frac{m(1-m)\beta}{1-2m}, \quad 1 - 2m = p, \quad 2\beta - 1 = \epsilon \tag{1},$$

the same as those employed with D_{11} in § 22, a new relation

$$-p^3 + \epsilon^2(\epsilon - 2)p^2 + \epsilon(\epsilon^2 - \epsilon + 1)p + \epsilon^3 = 0. \tag{2},$$

representing a C_5 in (p, ϵ); and putting

$$p = -\frac{1}{x+1}, \quad \epsilon = \frac{1}{y+1} \tag{3},$$

this C_5 reduces to a C_4 in (x, y) with deficiency $p = 2$,

$$y^3 - (x^2 + 2x - 2)y^2 - (x + 4)xy + x^2(x + 2) = 0 \tag{4},$$

and this with $y = (q + 1)x$, as in L.M.S., 27, p. 463, becomes

$$(q+1)^2 x^2 - (q^3 + q^2 - 2q - 1)x - 2q^2 = 0 \tag{5},$$

$$x = \frac{q^3 + q^2 - 2q - 1 + \sqrt{Q}}{2(q+1)^2} \tag{6},$$

$$x + 1 = \frac{q^3 + 3q^2 + 2q + 1 + \sqrt{Q}}{2(q+1)^2} \tag{7}.$$

$$Q = q^6 + 2q^5 + 5q^4 + 10q^3 + 10q^2 + 4q + 1 \tag{8},$$

$$\frac{1}{x+1} = \frac{q^3 + 3q^2 + 2q + 1 - \sqrt{Q}}{2q^3} = -p \tag{9}.$$

$$y = \frac{q^3 + q^2 - 2q - 1 + \sqrt{Q}}{2(q+1)} \tag{10},$$

$$\frac{1}{y+1} = \frac{-q^3 - q^2 - 0 - 1 + \sqrt{Q}}{2q(q^2 + q + 1)} = \epsilon \tag{11}.$$

Now the p employed in (2), § 10, distinguished here as p_9, is given by

$$x = y(1 - z), \quad z - y = \frac{z^2}{p} \tag{12},$$

returning for a moment to the original x and y of § 10, so that

$$p_9 = \frac{z}{x - y} = \frac{(y - x)^2}{y(y - x - y^2)}$$

$$= \frac{s(2v) - s(v)}{s(5v) - s(v)} = \frac{s(2v) - s(v)}{s(4v) - s(v)}$$

$$= \frac{\dfrac{m^2\alpha}{\alpha - m} - \dfrac{m^2\alpha^2}{(\alpha - m)^2}}{\dfrac{m^2\alpha}{\alpha - m} - \dfrac{m^2\{(1 - 2m)\alpha - m(1 - m)\}^2}{(1 - 2m)^2(\alpha - m)^2}}$$

$$= \frac{-(1 - 2m)^2\alpha}{(1 - 2m)\alpha - m(1 - m)^2} = \frac{-(1 - 2m)\beta}{\beta - 1 + m}.$$

$$= \frac{-p(\epsilon + 1)}{\epsilon - p} \qquad (13);$$

and

$$\epsilon - p = \frac{(q + 1)(3q^3 + 3q^2 + 2q + 1 - \sqrt{Q})}{2q^3(q^2 + q + 1)} \qquad (14),$$

$$\frac{1}{\epsilon - p} = \frac{3q^3 + 3q^2 + 2q + 1 + \sqrt{Q}}{4(q + 1)^2} \qquad (15),$$

$$\epsilon + 1 = \frac{q^3 + q^2 + 2q - 1 + \sqrt{Q}}{2q(q^2 + q + 1)} = 2\beta \qquad (16),$$

$$\frac{\epsilon + 1}{\epsilon - p} = \frac{q^3 + q^2 + 2q + 1 + \sqrt{Q}}{2(q + 1)^2} \qquad (17),$$

$$p_9 = \frac{-p(\epsilon + 1)}{\epsilon - p} = \frac{q^3 + q^2 + 0 - 1 + \sqrt{Q}}{2q(q + 1)^2} \qquad (18).$$

This value of p_9, employed in the previous equations (7), § 10, for $\mu = 9$ will lead to the functions of the Second Stage for $\mu = 18$; the result, in a modified form, has been given already in the 'Archiv der Mathematik und Physik,' 3, Reihe, 4, p. 74.

Put

$$q = -\frac{1}{a + 1}, \quad Q = \frac{A}{(a + 1)^6}, \quad A = a^6 + 2 + 5 + 10 + 10 + 4 + 1 \qquad (19),$$

to agree with the results in the 'Archiv,' and in the modified form of § 28

$$I = \int \frac{Py^2 - Q}{y} \frac{dy}{\sqrt{Y_1 Y_2}}$$

$$= \frac{2}{9}\cos^{-1}\frac{y^3 + A_1y^2 + A_2y + A_3}{y^2}\sqrt{\tfrac{1}{2}Y_1}$$

$$= \frac{2}{9}\sin^{-1}\frac{y^3 - A_1y^2 + A_2y - A_3}{y^2}\sqrt{\tfrac{1}{2}Y_2} \qquad (20),$$

$$\begin{aligned} \mathrm{Y}_1 &= (y + 1)(y^2 + \mathrm{C}_1 y + \mathrm{C}_2) \\ \mathrm{Y}_2 &= (y - 1)(y^2 - \mathrm{C}_1 y + \mathrm{C}_2) \end{aligned} \qquad (21).$$

$$\begin{aligned} \mathrm{C}_1 &= -\frac{1}{2a} \\ \mathrm{C}_2 &= \frac{1 - 2m}{2a} = \mathrm{Q} \end{aligned} \qquad (22),$$

$$\mathrm{A}_1 = \frac{2a^5 + 6 + 11 + 13 + 6 + 1 + (2a^2 + 4a + 3)\sqrt{\mathrm{A}}}{2(a + 1)^2(a^2 + a + 1)} \qquad (23).$$

$$\mathrm{A}_2 = \frac{-a^6 - 4 - 9 - 16 - 20 - 14 - 5 - (a^3 + 3 + 4 + 3)\sqrt{\mathrm{A}}}{2(a + 1)^4} \qquad (24).$$

$$\mathrm{A}_3 = \frac{\left\{\begin{matrix} -a^9 - 5 - 15 - 34 - 58 - 73 - 63 - 31 - 6 + 1 \\ + (-a^6 - 4 - 9 - 14 - 14 - 8 - 1)\sqrt{\mathrm{A}} \end{matrix}\right\}}{2(a + 1)^4(a^2 + a + 1)} \qquad (25),$$

$$\mathrm{C}_1 = \frac{\left\{\begin{matrix} -1 - 5 - 11 - 6 + 14 + 31 + 29 + 17 + 6 + 1 \\ + (-1 - 4 - 5 - 6 - 6 - 4 - 1)\sqrt{\mathrm{A}} \end{matrix}\right\}}{4a^4(a + 1)^2(a^2 + a + 1)} \qquad (26),$$

$$\mathrm{C}_2 = \frac{\left\{\begin{matrix} 1 + 3 + 5 + 6 + 4 + 7 + 11 + 9 + 4 + 1 \\ + (1 + 2 + 1 - 2 - 2 - 2 - 1)\sqrt{\mathrm{A}} \end{matrix}\right\}}{4a^4(a + 1)^2(a^2 + a + 1)} \qquad (27).$$

Expressed in terms of a, the parameters employed by KIEPERT ('Math. Ann.,' 32. p. 128) are given by

$$\tau_{18} + 2 = \frac{1}{\xi_3} = \frac{a^3 + a^2 - 2a - 1 + \sqrt{\mathrm{A}}}{2a(a + 1)} \qquad (28),$$

τ_{18} being GIERSTER's τ ('Math. Ann.,' 14. p. 540),

$$\tau_{18} = \xi_6 = \frac{a^3 - 3a^2 - 6a - 1 + \sqrt{\mathrm{A}}}{2a(a + 1)} \qquad (29).$$

$$\xi_3 = \frac{-a^3 - a^2 + 2a + 1 + \sqrt{\mathrm{A}}}{4a(a + 1)} \qquad (30),$$

$$\xi_1 = \frac{a^3 + 3a^2 + 0 - 1 - \sqrt{\mathrm{A}}}{2a(a + 1)} \qquad (31).$$

$$\xi_2 = \frac{a^3 + 3a^2 + 0 - 1 - \sqrt{\mathrm{A}}}{2(a^3 - 3a - 1)} \qquad (32),$$

$$\frac{\xi_1}{\xi_2} = \frac{a^3 - 3a - 1}{a(a + 1)} \qquad (33).$$

$$\xi_4 = \frac{-a^3 - 9a^2 - 6a + 1 + 3\sqrt{A}}{2(a^3 - 3a - 1)} \qquad (34),$$

$$\xi_5 = \frac{a^3 - a^2 - 4a - 1 + \sqrt{A}}{2a(a+1)} \qquad (35),$$

$$\xi_6 = \frac{a^3 - 3a^2 - 6a - 1 + \sqrt{A}}{2a(a+1)} \qquad (36),$$

$$\mathrm{L}\,(2)^{24} = \qquad (37).$$

Then for the section-values

$$\kappa^2\mathrm{tn}^4\tfrac{2}{9}\mathrm{K}' = a_1 = \frac{\left\{\begin{array}{c} q^9 + 1 + 5 + 28 + 66 + 83 + 73 + 29 + 8 + 1 \\ + (q^6 - 2 - 11 - 18 - 14 - 6 - 1)\sqrt{Q} \end{array}\right\}}{2(q+1)^7(q^2+q+1)} \qquad (38),$$

$$\kappa^2\mathrm{tn}^4\tfrac{7}{9}\mathrm{K}' = \frac{1}{a_1} \qquad (39),$$

$$\kappa^2\mathrm{tn}^4\tfrac{4}{9}\mathrm{K}' = a_2 = \frac{q^6 + 4 + 9 + 16 + 14 + 6 + 1 \pm (q^3 + 3q^2 + 4q + 1)\sqrt{Q}}{2(q+1)(q^2+q+1)} \qquad (40),$$

$$\kappa^2\mathrm{tn}^4\tfrac{5}{9}\mathrm{K}' = \frac{1}{a_2} = \frac{1 + 4 + 9 + 16 + 14 + 6 + 1 - (1 + 3 + 4 + 1)\sqrt{Q}}{2q^3(q^3 - 3q - 1)} \qquad (41).$$

$$1 - \frac{m}{\alpha} = \left(\frac{q^3 + q^2 + 0 - 1 - \sqrt{Q}}{2q}\right)^2 \qquad (42),$$

$$\alpha = \frac{-q^9-3-5-6-4-7-11-9-4-1+(-q^6-2-1+2+2+2+1)\sqrt{Q}}{16q^4(q+1)^2(q^2+q+1)} \qquad (43).$$

$$\gamma_9 = 0, \quad \lambda = \frac{\gamma_5}{\gamma_4} = \frac{y-x}{y} = z = p(1-p) \qquad (44),$$

$$p = p_9 = \frac{q^3 + q^2 + 0 - 1 + \sqrt{Q}}{2q(q+1)^2} \qquad (45).$$

$$\left(\frac{\Theta\frac{2}{9}\mathrm{K}'}{\Theta 0}\right)^9 = c_1^{\,9} = \frac{\alpha^3}{z}\left(1 - \frac{m}{\alpha}\right) \qquad (46),$$

$$\left(\frac{\Theta\frac{4}{9}\mathrm{K}'}{\Theta 0}\right)^9 = c_2^{\,9} = \frac{\alpha^3}{z^4}\left(\frac{\alpha}{\alpha - m}\right)^3 \qquad (47),$$

$$\begin{aligned}\frac{\Theta\frac{2}{3}\mathrm{K}'}{\Theta 0} = c_3 &= \frac{q^3 + q^2 - 2q - 1 + \sqrt{Q}}{-4q(q+1)} \\ &= \frac{-1}{\tau_{18} + 2} = \frac{(b^2 - 1)^{\frac{1}{2}}}{2}\end{aligned} \qquad (48),$$

from the table for $\mu = 6$, § 29.

33. $\mu = 22, \quad f = \frac{r}{11}$.

The relation $D_{11} = 0$ in (21), § 22, gives a C_8 in (p, ϵ); but putting

$$\epsilon = \frac{1}{x+1}, \quad p = \frac{1}{y+1} \tag{1}$$

it becomes

$$(x+1)^2 y^5 + (4x^2 + 9x + 4) y^4 - (x+2)(3x^2 - 3x - 2) y^3$$
$$+ x(x+2)(x^2 - 7x - 2) y^2 + 4x^3 (x+2) y - x^4 (x+2) = 0 \tag{2}$$

a C_7 in (x, y) with triple point at the origin; so putting $y = -sx$,

$$s^5 x^4 + (2s^3 - 4s^2 - 3s - 1) s^2 x^3 + (s^5 - 9s^4 - 3s^3 + 5s^2 + 4s + 1) x^2$$
$$- 2(2s^4 - 4s^3 - 8s^2 - 4s - 1) x + 4s^2 (s+1) = 0 \tag{3}$$

a quartic for x, which can be resolved

$$[2s^3 x^2 + (2s^3 - 4s^2 - 3s - 1) x - 2(2s^2 + 2s + 1)]^2 = \{(s-1) x - 2\}^2 \mathrm{S} \tag{4}$$

$$\mathrm{S} = 4s(s+1)^2 + 1 \tag{5}$$

Writing it

$$[2sx(s^2 x - 2s - 1) + (2s^2 + 2s + 1)\{(s-1)x - 2\}]^2$$
$$= \{(s-1)x - 2\}^2 \mathrm{S} \tag{6}$$

and putting

$$\frac{2sx(s^2 x - 2s - 1)}{(s-1)x - 2} = -2s^2(t+1) \tag{7}$$

$$(-2s^2 t + 2s + 1)^2 = \mathrm{S}, \quad t = \frac{2s + 1 + \sqrt{\mathrm{S}}}{2s^2} \tag{8}$$

so that, with

$$u = \int^{\infty} \frac{ds}{\sqrt{\mathrm{S}}}, \quad s = s(u) \tag{9}$$

$$t = s(u - \tfrac{2}{5}\omega) \tag{10}$$

This is the elliptic integral of which the ikosahedron irrationality η of KLEIN is unity; it is curious also that this integral occurs as one of ABEL's numerical exercises ('Œuvres,' 1, p. 142).

Then to connect up with $\mu = 11$, as in L.M.S., 27, p. 469,

$$1 + c_{11} = p_{11} = \frac{s(2v) - s(v)}{s(5v) - s(v)} = \frac{(x-y)^2}{(y+1)(xy - 2x + 2y)} \tag{11}$$

after reduction; and

$$\sqrt{\mathrm{C}} = 1 - 2z_{11}, \quad z_{11} = \frac{s(3v) - s(2v)}{s(3v) - s(v)} \tag{12}$$

reducing to

$$z_{11} = \frac{-(y+2)(x-y)}{(y+1)(xy-2x+2y)}, \quad \frac{z_{11}}{p_{11}} = \frac{y+2}{y-x} \qquad (13),$$

where x and y are given in terms of s by equation (64).

Substituting these values of c and $\sqrt{C}$ in terms of x and y in the expression for $I(v)$ in (42), § 11, for $\mu = 11$, we obtain a result applicable to $\mu = 22$, and associated functions of the Second Stage.

34. The next two cases of $\mu = 13$, 26 and $\mu = 15$, 30 still present analytical difficulties not yet surmounted, although $\mu = 30$ could be treated by the trisection method of § 6, applied to $\mu = 10$.

35. When μ is of the form $4n$, so that

$$v = \omega_1 + \frac{2r+1}{2n}\omega_3, \quad \text{or} \quad K + \frac{2r+1}{2n}K'i, \quad f = \frac{2r+1}{2n} \qquad (1),$$

then $\frac{1}{2}\mu v = 2nv$ is congruent to ω_3, as in Case III., and $I(v)$ is given by

$$\begin{aligned} I(v) &= \frac{1}{2n}\cos^{-1}\frac{P_1 s^{n-1} + P_2 s^{n-2} + \ldots + P_n}{(\sigma - s)^n}\sqrt{(s-s_3)} \\ &= \frac{1}{2n}\sin^{-1}\frac{s^{n-1} + Q_2 s^{n-2} + \ldots + Q_n}{(\sigma - s)^n}\sqrt{(s_1 - s \,.\, s_2 - s)} \\ &= \int_s^{s_2} \frac{-P(v)(\sigma - s) + \frac{1}{2}\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}} \qquad (2), \end{aligned}$$

$$s_1 > \sigma > s_2 > s > s_3 \qquad (3).$$

Here the degree is halved by putting

$$s - s_3 = x^2 \qquad (4),$$

so that

$$\int \frac{ds}{\sqrt{S}} = \int \frac{dx}{\sqrt{X_1 X_2}} \qquad (5),$$

and

$$\begin{array}{ll|lr} & \tfrac{1}{4}\mu \text{ odd} & \tfrac{1}{4}\mu \text{ even} & \\ X_1 &= C_0 + C_1 x - x^2 & X_1 = C_0 + C_1 x + x^2 & \\ &= (x_1 - x)(x_2 + x) & \quad = (x_1 + x)(x_2 + x) & (6), \\ X_2 &= C_0 - C_1 x - x^2 & X_2 = C_0 - C_1 x + x^2 & \\ &= (x_1 + x)(x_2 - x) & \quad = (x_1 - x)(x_2 - x) & (7), \end{array}$$

also

$$x_2 = \kappa x_1 \qquad (8),$$

and putting

$$\delta = \sigma - s_3 = (s_1 - s_3)\,\mathrm{dn}^2 fK' = x_1^2\,\mathrm{dn}^2 fK' \qquad (9),$$

and

$$\mathrm{P}(v) = x_1 \mathrm{zn} f \mathrm{K}' \tag{10},$$

$$\mathrm{I}(v) = \int_x^{x_2} \frac{-\mathrm{P}(v)(\delta - x^2) + \frac{1}{2}\sqrt{-\Sigma}}{\delta - x^2} \frac{dx}{\sqrt{\mathrm{X}_1 \mathrm{X}_2}}$$

$$= \frac{1}{n} \cos^{-1} \frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{n-1} x^{n-1}}{(\delta - x^2)^{\frac{1}{2}n}} \sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{n} \sin^{-1} \frac{\mathrm{R}_0 - \mathrm{R}_1 x + \ldots + (-1)^{n-1} \mathrm{R}_{n-1} x^{n-1}}{(\delta - x^2)^{\frac{1}{2}n}} \sqrt{\tfrac{1}{2}\mathrm{X}_2} \tag{11}.$$

Thus for the vector of the corresponding herpolhode

$$\mathrm{M}\frac{\rho}{k} = \sqrt{(\delta - x^2)}, \quad x = x_2 \mathrm{sn}(\mathrm{K} - mt) \tag{12},$$

$$\mathrm{M}\frac{\rho}{k} e^{(pt-\varpi)i} = \{(\mathrm{R}_0 + \mathrm{R}_1 x + \ldots)\sqrt{\tfrac{1}{2}\mathrm{X}_1} + i(\mathrm{R}_0 - \mathrm{R}_1 x + \ldots)\sqrt{\tfrac{1}{2}\mathrm{X}_2}\}^{\frac{1}{n}} \tag{13},$$

and in the associated hodograph described by the axis of a top

$$\tfrac{1}{2}\mathrm{M}^2 \sin\theta e^{(pt-\psi)i}$$
$$= \{(\mathrm{A}_0 + \mathrm{A}_1 x + \ldots \mathrm{A}_{2n-1} x^{2n-1})\sqrt{\tfrac{1}{2}\mathrm{X}_1} + i(\mathrm{A}_0 - \mathrm{A}_1 x + \ldots - \mathrm{A}_{2n-1} x^{2n-1})\sqrt{\tfrac{1}{2}\mathrm{X}_2}\}^{\frac{1}{n}} \tag{14}.$$

36. With $\mu = 4n$, the resolution of T into factors for the spherical catenary must be such that T_1 and T_2 are quadratics in t^2; and thus

$$\mathrm{T}_1 = -\left\{k - 1 \pm 2\sqrt{\frac{k-h}{k+h}}\, t + (k+1)t^2\right\} \tag{15},$$

$$\mathrm{T}_2 = \left\{k + 1 \pm 2\sqrt{\frac{k+h}{k-h}}\, t + (k-1)t^2\right\} \tag{16},$$

$$\mathrm{T}_1 = -(k-1)^2 + 2\left(2\frac{k-h}{k+h} - k^2 + 1\right)t^2 - (k+1)^2 t^4 \tag{17},$$

$$\mathrm{T}_2 = (k+1)^2 - 2\left(2\frac{k+h}{k-h} - k^2 + 1\right)t^2 + (k-1)^2 t^4 \tag{18}.$$

37. The simplest cases may be cited before proceeding to the general theory.

$$\mu = 4, \quad v = \mathrm{K} + \tfrac{1}{2}\mathrm{K}'i \qquad (1),$$

$$\mathrm{I}(v) = \int_x^\kappa \frac{\tfrac{1}{2}(1-\kappa)(\kappa+x^2)}{\kappa - x^2} \frac{dx}{\sqrt{(1-x^2)(\kappa^2-x^2)}}$$

$$= \cos^{-1}\sqrt{\frac{(1-x)(\kappa+x)}{2(\kappa-x^2)}} = \sin^{-1}\sqrt{\frac{(1+x)(\kappa-x)}{2(\kappa-x^2)}} \qquad (2);$$

and in the associated dynamical applications

$$\mathrm{M}\frac{\rho}{k} = \sqrt{(\kappa - x^2)}, \quad x = \kappa\,\mathrm{sn}(\mathrm{K} - mt) \qquad (3),$$

$$\mathrm{M}\frac{\rho}{k}e^{(pt-\varpi)i} = \sqrt{\tfrac{1}{2}(1-x)(\kappa+x)} + i\sqrt{\tfrac{1}{2}(1+x)(\kappa-x)} \qquad (4),$$

reducing to HALPHEN's algebraical herpolhode when the secular term pt is cancelled by putting $p = 0$.

In the associated motion of the top (' Annals of Mathematics,' vol. 5)

$$\tfrac{1}{2}\mathrm{M}^2 \sin\theta e^{(pt-\psi)i} = \left(\frac{p}{m} - \frac{1-\kappa}{2} - x\right)\sqrt{\tfrac{1}{2}(1-x)(\kappa+x)}$$

$$+ i\left(\frac{p}{m} - \frac{1-\kappa}{2} + x\right)\sqrt{\tfrac{1}{2}(1+x)(\kappa-x)} \qquad (5).$$

But the quadric transformation

$$z = \frac{(1-\kappa)x}{\kappa - x^2} = \mathrm{cn}[(1+\kappa)mt, \gamma], \quad \gamma = \frac{2\sqrt{\kappa}}{1+\kappa} \qquad (6),$$

makes

$$\mathrm{I}(v) = \tfrac{1}{2}\int_z^1 \frac{dz}{\sqrt{(1-z^2)}} = \cos^{-1}\sqrt{\frac{1+z}{2}} = \sin^{-1}\sqrt{\frac{1-z}{2}} \qquad (7),$$

thus effecting the identification with the result, p. 147, 'Œuvres,' 2, when the sign of ABEL's R is changed so as to obtain the circular form.

For the division-values

$$\mathrm{sn}\,\tfrac{1}{2}\mathrm{K}' = \sqrt{\frac{1}{1+\kappa}}, \quad \mathrm{cn}\,\tfrac{1}{2}\mathrm{K}' = \sqrt{\frac{\kappa}{1+\kappa}}, \quad \mathrm{dn}\,\tfrac{1}{2}\mathrm{K}' = \sqrt{\kappa} \qquad (8);$$

$$\mathrm{zn}\,\tfrac{1}{2}\mathrm{K}' = \mathrm{zd}\,\tfrac{1}{2}\mathrm{K}' = \tfrac{1}{2}(1-\kappa), \quad \mathrm{zs}\,\tfrac{1}{2}\mathrm{K}' = -\,\mathrm{zc}\,\tfrac{1}{2}\mathrm{K}' = \tfrac{1}{2}(1+\kappa) \qquad (9);$$

$$\frac{\Theta\,\tfrac{1}{2}\mathrm{K}'}{\Theta 0} = \frac{(1+\kappa)^{\frac{1}{4}}}{2^{\frac{1}{4}}\kappa^{\frac{1}{8}}}, \quad \frac{\mathrm{H}\,\tfrac{1}{2}\mathrm{K}'}{\Theta 0} = \frac{(1-\kappa)^{\frac{1}{4}}}{2^{\frac{1}{4}}\kappa^{\frac{1}{8}}}, \quad \frac{\mathrm{H}\,\tfrac{1}{2}\mathrm{K}'}{\mathrm{HK}'} = \frac{\kappa^{\frac{1}{8}}}{2^{\frac{1}{4}}(1+\kappa)^{\frac{1}{4}}} \qquad (10).$$

38.
$$\mu = 8, \quad v = \mathrm{K} + \tfrac{1,3}{4}\,\mathrm{K}'i,$$

$$\mathrm{I}\,(v) = \int_x^\kappa \frac{-\,\mathrm{P}\,(v)\,(\delta - x^2) + \mathrm{Q}\,(v)}{\delta - x^2}\,\frac{dx}{\sqrt{\mathrm{X}_1\mathrm{X}_2}}$$

$$= \frac{1}{2}\cos^{-1}\frac{\frac{1}{2}(1+\alpha)^2 - x}{\delta - x^2}\sqrt{\tfrac{1}{2}(1+x)(\kappa+x)}$$

$$= \frac{1}{2}\sin^{-1}\frac{\frac{1}{2}(1+\alpha)^2 + x}{\delta - x^2}\sqrt{\tfrac{1}{2}(1-x)(\kappa-x)} \qquad (1),$$

where the octahedron irrationality $o = \sqrt{\kappa}$ is given in terms of α by

$$o = \tfrac{1}{2}\left(\frac{1}{\alpha} - \alpha\right) \qquad (2),$$

unchanged by the substitution $\left(\alpha, -\dfrac{1}{\alpha}\right)$, which interchanges $f = \frac{1}{4}$ and $\frac{3}{4}$.

Also, in the region $\sqrt{2} > \alpha > \sqrt{2} - 1, \quad f = \frac{1}{4}$,

$$\delta = \frac{(1+\alpha)^3(1-\alpha)}{4\alpha} = \mathrm{dn}^2\tfrac{1}{4}\mathrm{K}' \qquad (3),$$

$$\mathrm{P}\,(v) = \frac{(1 + 2\alpha + 3\alpha^2)(-1 + 2\alpha + \alpha^2)}{16\alpha^2} = \mathrm{zn}\,\tfrac{1}{4}\mathrm{K}' \qquad (4)$$

$$\mathrm{Q}\,(v) = \tfrac{1}{2}\sqrt{-\Sigma} = \frac{(1+\alpha)^2(1-\alpha^4)(-1+2\alpha+\alpha^2)}{16\alpha^3}$$

$$= \kappa'^2\mathrm{sn}\,\tfrac{1}{4}\mathrm{K}'\,\mathrm{cn}\,\tfrac{1}{4}\mathrm{K}'\,\mathrm{dn}\,\tfrac{1}{4}\mathrm{K}' \qquad (5).$$

The result is obtained by putting

$$s\,(\omega) - s\,(4v) = 0, \quad \alpha = \frac{m\,(1-m)}{1-2m}, \quad \beta = 1 \qquad (6);$$

and the values of s_1, s_2, s_3 are rationalised by putting

$$m = \frac{1+\alpha}{1+2\alpha-\alpha^2}, \quad 1 - 8\,(1-2m)\,\alpha = \left(\frac{-1+2\alpha+\alpha^2}{1+2\alpha-\alpha^2}\right)^2 \qquad (7).$$

To normalise the results, a homogeneity factor M is introduced, such that

$$s - s_3 = \mathrm{M}^2x, \quad \mathrm{M} = \sqrt{(s_1 - s_3)} = \frac{2\alpha^2}{(1+\alpha)(1+2\alpha-\alpha^2)} \qquad (8).$$

The division-values are shown in tabular form, and the numerical test of $\alpha = \frac{1}{2}(\sqrt{5} - 1)$ is required for $\mu = 15$.

Region.	A. $\infty > a > \sqrt{2}+1.$	B. $\sqrt{2}+1 > a > 1.$	C. $1 > a > \sqrt{2}-1.$
$\frac{1}{4}\left(\frac{1}{a}-a\right)^2$	$\frac{1}{\kappa'}$	κ'	κ'
$\frac{(1+a^2)^2(-1+2a+a^2)(1+2a-a^2)}{16a^4}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	κ^2
$\frac{(1+a)^3(1-a)}{4a}$	$-\mathrm{sc}^2\,\frac{3}{4}\mathrm{K}$	$-\mathrm{cs}^2\,\frac{1}{4}\mathrm{K}$	$\mathrm{dn}^2\,\frac{1}{4}\mathrm{K}$
$\frac{(1-a)^3(1+a)}{4a^3}$	$-\mathrm{sc}^2\,\frac{1}{4}\mathrm{K}$	$-\mathrm{cs}^2\,\frac{3}{4}\mathrm{K}$	$\mathrm{dn}^2\,\frac{3}{4}\mathrm{K}$
$\frac{1}{2}\left(\frac{1}{a}-a\right)$	$-\mathrm{nd}\,\frac{1}{2}\mathrm{K}$	$-\mathrm{dn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{dn}\,\frac{1}{2}\mathrm{K}$
$\frac{(1+2a+3a^2)(-1+2a+a^2)}{16a^2}$	$\frac{1}{\kappa'}\,\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{4}\mathrm{K}$
$\frac{(3-2a+a^2)(-1+2a+a^2)}{16a^2}$	$\frac{1}{\kappa'}\,\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$\mathrm{zn}\,\frac{3}{4}\mathrm{K}$
$\frac{1}{8}\left(\frac{1}{a}+a\right)^2$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$
$\frac{a^{\frac{1}{16}}(1+a)^{\frac{1}{16}}(1-a)^{\frac{9}{16}}}{(1+a^2)^{\frac{1}{8}}(-1+2a+a^2)^{\frac{1}{4}}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{a^{\frac{9}{16}}(1+a)^{\frac{9}{16}}(1-a)^{\frac{1}{16}}}{(1+a^2)^{\frac{3}{8}}(-1+2a+a^2)^{\frac{1}{4}}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{\left(\frac{1}{a}+a\right)^{\frac{1}{4}}}{\left(\frac{1}{a}-a\right)^{\frac{3}{8}}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta 0}$
Test by a	$\sqrt{5}+2$	$\frac{1}{2}(\sqrt{5}+1)$	$\frac{1}{2}(\sqrt{5}-1)$

D. $\sqrt{2}-1>a>0.$	E. $0>a>-\sqrt{2}+1.$	F. $-\sqrt{2}+1>a>-1.$	G. $-1>a>-\sqrt{2}-1.$	H. $-\sqrt{2}-1>a>-\infty.$
$\frac{1}{\kappa'}$	$\frac{1}{\kappa'}$	κ'	κ'	$\frac{1}{\kappa'}$
$-\frac{\kappa^2}{\kappa'^2}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	κ^2	$-\frac{\kappa^2}{\kappa'^2}$
$\mathrm{nd}^2\,\frac{1}{4}\mathrm{K}$	$-\mathrm{sc}^2\,\frac{3}{4}\mathrm{K}$	$-\mathrm{cs}^2\,\frac{3}{4}\mathrm{K}$	$\mathrm{dn}^2\,\frac{3}{4}\mathrm{K}$	$\mathrm{nd}^2\,\frac{3}{4}\mathrm{K}$
$\mathrm{nd}^2\,\frac{3}{4}\mathrm{K}$	$-\mathrm{sc}^2\,\frac{1}{4}\mathrm{K}$	$-\mathrm{cs}^2\,\frac{1}{4}\mathrm{K}$	$\mathrm{dn}^2\,\frac{1}{4}\mathrm{K}$	$\mathrm{nd}^2\,\frac{1}{4}\mathrm{K}$
$\mathrm{nd}\,\frac{1}{2}\mathrm{K}$	$-\mathrm{nd}\,\frac{1}{2}\mathrm{K}$	$-\mathrm{dn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{dn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{nd}\,\frac{1}{2}\mathrm{K}$
$-\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{3}{4}\mathrm{K}$	$-\frac{1}{\kappa'}\,\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$-\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$-\mathrm{zn}\,\frac{3}{4}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{4}\mathrm{K}$
$-\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{4}\mathrm{K}$	$-\frac{1}{\kappa'}\,\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$-\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$-\mathrm{zn}\,\frac{1}{4}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{3}{4}\mathrm{K}$
$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{2}\mathrm{K}$
$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$
$\sqrt{5}-2$	$-\sqrt{5}+2$	$-\frac{1}{2}(\sqrt{5}-1)$	$-\frac{1}{2}(\sqrt{5}+1)$	$-\sqrt{5}-2$

39. ABEL's form in 'Œuvres,' 1, p. 142, 2, p. 160, for $n = 2$ becomes the same as in 2, p. 148, by replacing his $x + \frac{1}{4}a$ by x, his a by $2(q + q')$, and b by $-\frac{1}{4}(q - q')^2$.

But now in our notation for $\mu = 8$, equation (14) § 25 becomes

$$Z_2 = [z + m(1 - m)]^2 - (1 - 2m)^2 z \tag{1}$$

and putting

$$m = \frac{1 + a}{1 + 2a - a^2}, \quad 1 - m = \frac{a(1 - a)}{1 + 2a - a^2}, \quad 1 - 8m + 8m^2 = \left(\frac{-1 + 2a + a^2}{1 + 2a - a^2}\right)^2 \tag{2}$$

and replacing z by $\frac{z}{M}$, where $M = 1 + 2a - a^2$, the roots of $Z = Z_1 Z_2 = 0$ are

$$z_0 = (1 + a)^2, \quad z_1 = a^2(1 + a)^2, \quad z_2 = (1 - a)^2, \quad z_3 = a^2(1 - a)^2 \tag{3}$$

A quadric transformation is required in ABEL's result to obtain our $I(r)$, namely, with the δ above, and the sign of Z_1 changed to obtain the circular form,

$$z_0 - z = \frac{(1 + a)^3(1 - a)(1 - x^2)}{\delta - x^2} \tag{4}$$

$$z_1 - z = \frac{(1 + a^2)(-1 + 2a + a^2)\delta}{\delta - x^2} \tag{5}$$

$$z_2 - z = \frac{(1 + a^2)(-1 + 2a + a^2)x^2}{\delta - x^2} \tag{6}$$

$$z - z_3 = \frac{4a^3(1 - a^2)^4(\kappa^2 - x^2)}{\delta - x^2} \tag{7}$$

and

$$Z_1 = (z_0 - z)(z - z_3) = \frac{4a^3(1 + a)^7(1 - a)^5(1 - x^2)(\kappa^2 - x^2)}{(\delta - x^2)^2} \tag{8}$$

$$Z_2 = (z_1 - z)(z_2 - z) = \frac{(1 + a^2)^2(-1 + 2a + a^2)^2\delta x^2}{(\delta - x^2)^2} \tag{9}$$

so that $\sqrt{Z_2}$ is rational.

Afterwards the degree is halved, as ABEL's integral is $2I(r)$.

A similar quadric transformation and halving of degree will be required for all even values of ABEL's n, to reduce his results to our form, involving heavy algebraical work.

ABEL's integral ('Œuvres,' 2, p. 148), changed into the circular form, can be utilised for the construction of an algebraical orbit described under the central attraction

$$P = \frac{\mu}{r^2} + \mu_0 + \mu_1 r \tag{10}$$

in the form

$$N \cos 2\theta = (u + q + 2q') \sqrt{(u^2 + 2qu - q^2 - 2qq')} \qquad (11),$$

$$N \sin 2\theta = (u + 2q + q') \sqrt{(- u^2 - 2q'u + 2qq' + q'^2)} \qquad (12),$$

$$N^2 = (q' - q)^3 (q' + q) \qquad (13),$$

so that the condition

$$\theta = \int \frac{u\,du}{\sqrt{U}} \qquad (14),$$

$$U = (u^2 + 2qu - q^2 - 2qq')(- u^2 - 2q'u + 2qq' + q'^2) \qquad (15),$$

is satisfied by taking

$$\frac{\mu}{h^2} = - q - q', \quad \frac{H}{h^2} = q^2 + q'^2, \quad \frac{\mu_0}{h^2} = - 3qq' (q + q')$$

$$\frac{\mu_1}{h^2} = qq' (q + 2q') (2q + q') \qquad (16).$$

A degenerate circular orbit $u = q$ is obtained by taking

$$q + q' = 0, \quad \mu = 0, \quad \mu_0 = 0, \quad H = 2q^2h^2,$$

$$P = h^2q^3, \quad v^2 = h^2q^2 \qquad (17).$$

But the integral in 'Œuvres,' 2, p. 147, putting $x + q = u$, requires $\mu = 0$. $\mu_0 = 0$. so that we merely obtain a conic for $P = \mu_1 r$.

40. $\mu = 12, \quad v = K + \frac{1 \text{ or } 5}{6} K'i, \quad f = \tfrac{1}{6} \text{ or } \tfrac{5}{6}.$

$$I(v) = \int_{x}^{x_2} \frac{- P(v)(\delta - x^2) + Q(v)}{a + a^2 + a^3 - x^2} \frac{dx}{\sqrt{X_1X_2}}$$

$$= \tfrac{1}{3} \cos^{-1} \frac{1 + a + a^2 + (1 - a)x - x^2}{(a + a^2 + a^3 - x^2)^{\frac{3}{2}}} \sqrt{\tfrac{1}{2} X_1}$$

$$= \tfrac{1}{3} \sin^{-1} \frac{1 + a + a^2 - (1 - a)x - x^2}{(a + a^2 + a^3 - x^2)^{\frac{3}{2}}} \sqrt{\tfrac{1}{2} X_2} \qquad (1),$$

$$X_1, X_2 = a^3 (1 + a + a^2) \pm \tfrac{1}{2} (1 + a)^3 (1 - a) x - x^2 \qquad (2),$$

$$P(v) = \tfrac{1}{12} (1 - a)(5 + 3a + 3a^2 + a^3) = x_1 \mathrm{zn} f K' \qquad (3),$$

$$Q(v) = \tfrac{1}{2} (1 - a^4)(a + a^2 + a^3) = x_1^3 \kappa'^2 \mathrm{sn} f K' \mathrm{cn} f K' \mathrm{dn} f K' \qquad (4).$$

results obtained by putting

$$s(\omega) - s(6v) = 0, \quad \alpha = \frac{m(1-m)(1-3m+3m^2)}{(1-2m)(1-2m+2m^2)}, \quad \beta = \frac{1-3m+3m^2}{1-2m+2m^2} \tag{5}$$

$$\mathrm{M} = m = \frac{1}{1-a} \tag{6}$$

Treated by the trisection of $\mu = 4$, putting $y = 0$ in equation (8), § 14,

$$3t^4 + (1-8x)t^3 + 6x^2t^2 - x^4 = 0 \tag{7}$$

and putting

$$t = \frac{x}{b},$$

$$x = \frac{b}{(b-1)^3(b+3)}, \quad t = \frac{1}{(b-1)^3(b+3)} \tag{8}$$

$$\tau_4 = \frac{1}{8x} = \frac{(b-1)^3(b+3)}{8b} \tag{9}$$

and

$$\tau_4 = \frac{(\tau-4)^3\tau}{8(\tau-3)}, \quad \tau = \tau_{12} \tag{10}$$

so that

$$b = \tau_{12} - 3 \tag{11}$$

and b is the equivalent of y in KLEIN-FRICKE, 'Modulfunctionen,' 1, p. 688, while

$$x = -\tau_{18} - 2 \tag{12}$$

$$2\tau_6 + 8 = (\tau_{18}+2)^3 = y^2 = (\tau_{12}-3)^2 \tag{13}$$

$$x^3 + y^2 = 1 \tag{14}$$

and GIERSTER'S

$$\tau_{12} = \frac{36}{y^2-9} = \frac{36}{-x^3-8} \tag{15}$$

Also

$$2\tau_4 = \left(\tfrac{1}{o} - o\right)^2 = \frac{(1+a)^6(1-a)^2}{4a^3(1+a+a^2)} \tag{16}$$

so that

$$b = \frac{1}{a} + 1 + a \tag{17}$$

In KIEPERT'S notation, 'Math. Ann.,' 32, p. 104.

$$\xi_1 = 1 - \frac{4}{b-1}, \text{ \&c.} \tag{18}$$

Putting

$$\mathrm{A} = (1+a^2)(1+4a+a^2) \tag{19}$$

the section values are shown in the table :—

	$0 < a < 1.$	$1 < a < \infty.$
$\frac{(1+a)^3(1-a)}{2a\sqrt{(a+a^2+a^3)}}$	$\frac{1}{o} - o$	$-\frac{1}{o} + o$
$\frac{(1+a^2)\sqrt{A}}{2a\sqrt{(a+a^2+a^3)}}$	$\frac{1}{o} + o$	$\frac{1}{o} + o$
$\frac{-(1+a)^3(1-a)+(1+a^2)\sqrt{A}}{(1+a)^3(1-a)+(1+a^2)\sqrt{A}}$	κ	$\frac{1}{\kappa}$
a	$\frac{\mathrm{dn}\,\frac{5}{6}K'}{o}$	$\frac{\mathrm{dn}\,\frac{1}{6}K'}{o}$
$\frac{(1+a)\sqrt{1+4a+a^2}-(1-a)\sqrt{1+a^2}}{(1+a)\sqrt{1+4a+a^2}+(1-a)\sqrt{1+a^2}}$	$\mathrm{dn}\,\frac{1}{3}K'$	$\mathrm{nd}\,\frac{1}{3}K'$
$\frac{(1+a)\sqrt{1+4a+a^2}-(1-a)\sqrt{1+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{cn}\,\frac{1}{3}K'$	$\mathrm{sn}\,\frac{2}{3}K'$
$\frac{(1+a)\sqrt{1+4a+a^2}+(1-a)\sqrt{1+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{sn}\,\frac{2}{3}K'$	$\mathrm{cn}\,\frac{1}{3}K'$
$\frac{(1+a)\sqrt{1+a^2}-(1-a)\sqrt{1+4a+a^2}}{(1+a)\sqrt{1+a^2}+(1-a)\sqrt{1+4a+a^2}}$	$\mathrm{dn}\,\frac{2}{3}K'$	$\mathrm{nd}\,\frac{2}{3}K'$
$\frac{(1+a)\sqrt{1+a^2}-(1-a)\sqrt{1+4a+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{cn}\,\frac{2}{3}K'$	$\mathrm{sn}\,\frac{1}{3}K'$
$\frac{(1+a)\sqrt{1+a^2}+(1-a)\sqrt{1+4a+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{sn}\,\frac{1}{3}K'$	$\mathrm{cn}\,\frac{2}{3}K'$
$\frac{-(1+a)^3(1-a)+(1+a^2)\sqrt{A}}{4a\sqrt{(a+a^2+a^3)}}$	$\mathrm{dn}\,\frac{1}{2}K'$	$\mathrm{nd}\,\frac{1}{2}K'$
$\frac{(1-a)(5+3a+3a^2+a^3)}{12a\sqrt{(a+a^2+a^3)}}$	$\frac{\mathrm{zn}\,\frac{1}{6}K'}{o}$	$\frac{\mathrm{zn}\,\frac{5}{6}K'}{o}$
$\frac{(1-a)(1+3a+3a^2+5a^3)}{12a\sqrt{(a+a^2+a^3)}}$	$\frac{\mathrm{zn}\,\frac{5}{6}K'}{o}$	$\frac{\mathrm{zn}\,\frac{1}{6}K'}{o}$
$\frac{(1+a+a^2)^2}{3a\sqrt{(a+a^2+a^3)}}$	$\frac{\mathrm{zs}\,\frac{1}{3}K'}{o}$	$\frac{\mathrm{zs}\,\frac{1}{3}K'}{o}$
$\frac{(1+a)^3(1-a)}{4a\sqrt{(a+a^2+a^3)}}$	$\frac{\mathrm{zn}\,\frac{1}{2}K'}{o}$	$-\frac{\mathrm{zn}\,\frac{1}{2}K'}{o}$
$\frac{a^{22}(a+a^2+a^3)^3}{(1-a)^6(1+a)^{10}(1+a^2)^8}$	$\left(\frac{\Theta\,\frac{1}{6}K'}{HK'}\right)^{24}$	$\left(\frac{\Theta\,\frac{5}{6}K'}{HK'}\right)^{24}$
$\frac{(a+a^2+a^3)^3}{a^2(1-a)^6(1+a)^{10}(1+a^2)^8}$	$\left(\frac{\Theta\,\frac{5}{6}K'}{HK'}\right)^{24}$	$\left(\frac{\Theta\,\frac{1}{6}K'}{HK'}\right)^{24}$

41. In the general case a distinction must be made between $\frac{1}{4}\mu$ odd or even.

I. $$\tfrac{1}{4}\mu = 2n + 1, \quad \mu = 8n + 4.$$

$$v = \mathrm{K} + \frac{2r+1}{4n+2}\mathrm{K}'i, \quad \text{or} \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \frac{2r+1}{4n+2} \qquad (1):$$

$$\mathrm{X}_1 = \mathrm{C}_0 + 2\mathrm{C}_1 x - x^2 = (x_1 - x)(x_2 + x) \qquad (2),$$

$$\mathrm{X}_2 = \mathrm{C}_0 - 2\mathrm{C}_1 x - x^2 = (x_1 + x)(x_2 - x) \qquad (3),$$

$$x_2 = \kappa x_1 \qquad (4),$$

$$\mathrm{C}_0 = \sqrt{(s_1 - s_3)(s_2 - s_3)} = s(2n+1)v - s(\omega_3) \qquad (5),$$

$$2\mathrm{C}_1 = \sqrt{(s_1 - s_3)} - \sqrt{(s_2 - s_3)} = 2\mathrm{P}(2n+1)v \qquad (6),$$

$$2\mathrm{C}_0\mathrm{C}_1 = \tfrac{1}{2}i\wp'(2n+1)v = \mathrm{Q}(2n+1)v \qquad (7),$$

$$\left(\frac{1}{o} - o\right)^2 = \frac{\mathrm{C}_1{}^2}{\mathrm{C}_0} = \tfrac{1}{4}\mathrm{L}(4)^8 \qquad (8),$$

where o denotes the octahedron irrationality $\sqrt{\kappa}$; so that if the expressions are normalised by the homogeneity factor $\mathrm{M} = \sqrt{\mathrm{C}_0}$,

$$\mathrm{X}_1, \mathrm{X}_2 = 1 \pm \mathrm{C}x - x^2, \quad \mathrm{C} = \frac{1}{o} - o \qquad (9),$$

$$x_1 = \frac{1}{o}, \quad x_2 = o, \quad x = o\,\mathrm{sn}(\mathrm{K} - mt) \qquad (10);$$

$$\mathrm{dn}^2 f\mathrm{K}' = \frac{\sigma - s_3}{s_1 - s_3} = \frac{\mathrm{C}_0\mathrm{D}}{s_1 - s_3} = \kappa\mathrm{D} = o^2\mathrm{D}, \quad \mathrm{zn} f\mathrm{K}' = \frac{\mathrm{P}(v)}{\sqrt{(s_1 - s_3)}} = \frac{o\mathrm{P}(v)}{\mathrm{M}} \qquad (11).$$

Now

$$\mathrm{I}(v) = \int_x \frac{-\frac{\mathrm{P}(v)}{\mathrm{M}}(\delta - x^2) + \frac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\mathrm{D} - x^2}\frac{dx}{\sqrt{\mathrm{X}_1\mathrm{X}_2}}$$

$$= \frac{1}{2n+1}\cos^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n}x^{2n}}{(\mathrm{D} - x^2)^{n+\frac{1}{2}}}\sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{2n+1}\sin^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n}x^{2n}}{(\mathrm{D} - x^2)^{n+\frac{1}{2}}}\sqrt{\tfrac{1}{2}\mathrm{X}_2} \qquad (12).$$

$$\mathrm{R}_0{}^2\mathrm{C}_0 = \mathrm{D}^{2n+1},$$

$$\mathrm{R}_{2n-1} = -\{(2n+1)\mathrm{P}(v) - \mathrm{P}(2n+1)v\}\mathrm{R}_{2n}, \quad \mathrm{R}_{2n} = (-1)^n,$$

$$\frac{\mathrm{Q}(v)}{\mathrm{M}^3} = \frac{\kappa'^2 \mathrm{sn} f\mathrm{K}' \mathrm{cn} f\mathrm{K}' \mathrm{dn} f\mathrm{K}'}{\kappa^{\frac{3}{2}}} \qquad (13).$$

Next putting

$$\frac{\alpha - m}{\alpha - m + 1} = c^4 \qquad (14),$$

$$C_0^2 = (s_1 - s_3)(s_2 - s_3) = \frac{m^4\alpha^2(\alpha - m + 1)}{(\alpha - m)^3} = \frac{m^4\alpha^2}{(\alpha - m)^2}\frac{1}{c^4} \qquad (15),$$

and with

$$\delta = s(v) - s(\omega_3) = -\frac{m^2\alpha}{\alpha - m} \qquad (16),$$

$$\delta \text{ positive}, \quad v = \omega_1 + f\omega_3, \qquad \delta \text{ negative}, \quad v = f\omega_3.$$

$$C_0 = -\frac{m^2\alpha}{\alpha - m}\frac{1}{c^2} = \frac{\delta}{c^2} \qquad (17);$$

so that when normalised by $M = \sqrt{C_0}$, D becomes c^2 and R_0 becomes c^{2n+1}.

In the sequel it is convenient to put

$$c = \frac{a - b}{a + b} = \frac{\operatorname{dn} f K'}{o} = \frac{o}{\operatorname{dn}(1 - f)K'} \qquad (18),$$

$$p = 1 - 2m = \frac{a - b^2}{b(1 - a)} \qquad (19),$$

$$p - 1 = \frac{(a - b)(1 + b)}{b(1 - a)}, \quad p + 1 = \frac{(a + b)(1 - b)}{b(1 - a)} \qquad (20);$$

so that

$$\alpha - m = \frac{c^4}{1 - c^4} = \frac{(a - b)^4}{8ab(a^2 + b^2)} \qquad (21),$$

$$\begin{aligned}\alpha &= \frac{1 - p}{2} + \frac{c^4}{1 - c^4} = \frac{1 + c^4}{2(1 - c^4)} - \frac{p}{2} \\ &= \frac{a^4 + 6a^2b^2 + b^4}{8ab(a^2 + b^2)} - \frac{a - b^2}{2b(1 - a)} \\ &= -\frac{(a^2 - b^2)\{3a^2 + a^3 + (1 + 3a)b^2\}}{8ab(1 - a)(a^2 + b^2)} \qquad (22).\end{aligned}$$

$$\begin{aligned}\beta &= \frac{(1 - 2m)\alpha}{m(1 - m)} = -\frac{4p\alpha}{p^2 - 1} \\ &= \frac{(a - b^2)\{3a^2 + a^3 + (1 + 3a)b^2\}}{2a(1 - b^2)(a^2 + b^2)} \qquad (23),\end{aligned}$$

$$\epsilon = 2\beta - 1 = \frac{2a^3 + a^4 - (1 + 2a)b^4}{a(1 - b^2)(a^2 + b^2)} \qquad (24),$$

$$\epsilon - 1 = \frac{(1 + a)(a^2 - b^2)(a + b^2)}{a(1 - b^2)(a^2 + b^2)} \qquad (25),$$

$$\epsilon + 1 = \frac{\{3a^2 + a^3 + (1 + 3a)\,b^2\}\,(a - b^2)}{a\,(1 - b^2)\,(a^2 + b^2)} \qquad (26),$$

$$\delta = \frac{(a + b)^2\,(1 + b)^2\,\{3a^2 + a^3 + (1 + 3a)\,b^2\}}{(1 - a)^3\,b^2} \qquad (27),$$

$$\mathrm{C}_0 = \frac{(a + b)^4\,(1 + b)^2\,\{3a^2 + a^3 + (1 + 3a)\,b^2\}}{(1 - a)^3\,b^2\,(a - b)^2} \qquad (28).$$

$$\begin{aligned}\mathrm{C}_1{}^2 + 2\mathrm{C}_0 &= x_1{}^2 + x_2{}^2 = s_1 + s_2 - 2s_3 \\ &= m^2\,\frac{1 - 4\,(1 - 2m)\,\alpha - 8\alpha^2}{4\,(\alpha - m)^2} \qquad (29),\end{aligned}$$

$$\begin{aligned}\mathrm{C}'^2 = \frac{\mathrm{C}_1'^2}{\mathrm{C}_0} &= -\,2 - c^2\,\frac{1 - 4\,(1 - 2m)\,\alpha - 8\alpha^2}{4\alpha\,(\alpha - m)} \\ &= -\,2 - \frac{1 - c^4}{c^2}\left\{\frac{1}{4\alpha} - 1 - 2\,(\alpha - m)\right\} \\ &= \left(\frac{1}{c} - c\right)^2 + \left(\frac{1}{c^2} - c^2\right)\frac{-1}{4\alpha} \\ &= \frac{16a^2b^2}{(a^2 - b^2)^2} + \frac{8ab\,(a^2 + b^2)\,2ab\,(1 - a)\,(a^2 + b^2)}{(a^2 - b^2)^3\,\{3a^2 + a^3 + (1 + 3a)\,b^2\}} \\ &= \frac{64a^3b^2\,(a^3 - b^4)}{(a^2 - b^2)^3\,\{3a^2 + a^3 + (1 + 3a)\,b^2\}} \qquad (30),\end{aligned}$$

$$\mathrm{C}_1'^2 = \frac{16a^3\,(a^3 - b^4)\,(1 + b)^2}{(1 - a)^3\,(a - b)^6} \qquad (31),$$

$$\tfrac{1}{4}\left(\tfrac{1}{o} - o\right)^2 = \tfrac{1}{4}\,\mathrm{C}'^2 = \frac{16a^3b^2\,(a^3 - b^4)}{3a^8 + a^9 + 6\,(a^4 - a^5)\,b^4 - (1 + 3a)\,b^8 - 8a^3b^2\,(a^3 - b^4)} \qquad (32),$$

$$\tfrac{1}{4}\left(\tfrac{1}{o} + o\right)^2 = \frac{3a^8 + a^9 + 6\,(a^4 - a^5)\,b^4 - (1 + 3a)\,b^8 + 8a^3b^2\,(a^3 - b^4)}{3a^8 + a^9 + 6\,(a^4 - a^5)\,b^4 - (1 + 3a)\,b^8 - 8a^3b^2\,(a^3 - b^4)} \qquad (33).$$

Thus in the quadric transformation

$$\lambda = \frac{2}{\frac{1}{o} + o}, \quad \lambda' = \frac{\frac{1}{o} - o}{\frac{1}{o} + o} \qquad (34),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} - \lambda\right)^2 = \frac{64a^6b^4\,(a^3 - b^4)^2}{\mathrm{D}} \qquad (35),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} + \lambda\right)^2 = \frac{\{3a^8 + a^9 + 6\,(a^4 - a^5)\,b^4 - (1 + 3a)\,b^8\}^2}{\mathrm{D}} \qquad (36),$$

$$\mathrm{D} = (a^4 - b^4)^3\,\{(3a^2 + a^3)^2 - (1 + 3a)^2\,b^4\} \qquad (37),$$

involving powers of b^4 only; this will be useful in the sequel.

To connect up with the order $\mu' = 4n + 2 = \frac{1}{2}\mu$, denoting $p(\mu')$ by p',

$$\frac{p'+1}{p'-1} = \sqrt{\frac{s(\omega_3) - s(6v)}{s(\omega_3) - s(3v)}} = \frac{(1 - 2m + 2m^2)\beta - (1 - 3m + 3m^2)}{m(1-m)(2\beta - 1)}$$

$$= -\frac{(p^2+1)\epsilon - 2p^2}{(p^2-1)\epsilon} = -\frac{1+b^2}{1-b^2}\cdot\frac{(a-b^2)^2}{2a^3 + a^4 - (1+2a)b^4} \qquad (38),$$

$$p' = \frac{p^2 - \epsilon}{(\epsilon - 1)p^2} = -\frac{(1 + b^2 \,.\, a^2 - b^2)^2}{1 - b^4 \,.\, 2a^3 + a^4 - (1+2a)b^4} \qquad (39).$$

We have found

$$\frac{\operatorname{dn} f\mathrm{K}'}{o} = c = \frac{a-b}{a+b} = \frac{\Theta(1-f)\mathrm{K}'}{\Theta(f\mathrm{K}')} \qquad (40),$$

and proceeding with the series, writing f for $f\mathrm{K}'$,

$$\frac{\operatorname{dn} 3f}{o} = \frac{1-m}{m}c = \frac{1-b}{1+b} \qquad (41),$$

$$\frac{\operatorname{dn}(2r+1)f}{o} = \frac{\mathrm{N}_{2r+1}}{\mathrm{D}_{2r+1}}c \qquad (42).$$

Working these out with the assistance of the analysis given subsequently,

$$\frac{\operatorname{dn} 5f}{o} = \frac{a^3 - (1 + a - a^2)b^4 + ab(a - a^2 - a^3 + b^4)}{a^3 - (1 + a - a^2)b^4 - ab(a - a^2 - a^3 + b^4)} \qquad (43),$$

and generally

$$\frac{\operatorname{dn}(2r+1)f}{o} = \frac{\mathrm{E}_{2r+1} + b\mathrm{F}_{2r+1}}{\mathrm{E}_{2r+1} - b\mathrm{F}_{2r+1}} \qquad (44),$$

where F is a rational function of a and b^4, and E derivable from it by the substitution

$$\left(a, \frac{1}{a}\right)\left(b, \frac{1}{b}\right).$$

Thus for

$$\mu = 8n + 4, \quad \frac{\operatorname{dn}(2n+1)f}{o} = 1, \quad \mathrm{F}_{2n+1} = 0 \qquad (45).$$

The results in the sequel give

$$\mathrm{F}_7 = a^4(1 + a - 2a^2 - a^3) - (a + a^2 - a^3 - 3a^4)b^4 - a^2b^8 \qquad (46),$$

and therefore

$$\mathrm{E}_7 = a^5 - (3a^3 + a^4 - a^5 - a^6)b^4 + (1 + 2a - a^2 - a^3)b^8 \qquad (47),$$

$$\mathrm{F}_9 = a^6(3 + 0 - 3a^2 - a^3) - 3a^3(1 + 2a - 2a^2 - 2a^3)b^4$$
$$+ (1 + 3a + 0 - 6a^3 + 0 + 0 - a^6)b^8 + a^3b^{12} \qquad (48),$$

$$E_9 = a^6 + a^3(-1 + 0 + 0 - 6a^3 + 0 + 3a^5 + a^6)b^4$$
$$+ 3a^3(2 + 2a - 2a^2 - a^3)b^8 - (1 + 3a + 0 - 3a^3)b^{12} \qquad (49),$$

$$F_{11} = A_0 + A_1 b^4 + A_2 b^8 + A_3 b^{12} + A_4 b^{16} + A_5 b^{20} \qquad (50),$$

$$E_{11} = B_5 + B_4 b^4 + B_3 b^8 + B_2 b^{12} + B_1 b^{16} + B_0 b^{20} \qquad (51),$$

where the A's are found in § 45 (3), and the B's are derivable by the substitution $\left(a, \frac{1}{a}\right)$.

These expressions have been tested by putting

$$F_5 = 0, \quad b^4 = -a + a^2 + a^3 \qquad (52),$$

which makes

$$E_7 = a^2(1 - a^2)^3, \qquad F_7 = a(1 - a^2)^3 \qquad (53),$$

$$E_9 = a^3(1 + a)^5(1 - a)^4, \quad F_9 = a^2(1 + a)^5(1 - a)^7 \qquad (54),$$

$$E_{11} = a^5(1 + a)^8(1 - a)^7, \quad F_{11} = a^4(1 + a)^8(1 - a)^7 \qquad (55),$$

and thus verifies

$$\frac{\text{dn}7f}{o} = \frac{o}{\text{dn}3f} = \frac{1 + b}{1 - b}, \quad \frac{\text{dn}9f}{o} = \frac{\text{dn}11f}{o} = \frac{a + b}{a - b} \qquad (56).$$

42. $\mu = 20, \quad f = \frac{1, 3, 7, 9}{10}.$

$$b^4 = -a + a^2 + a^3, \quad a^3 - b^4 = a(1 - a) \qquad (1),$$

$$C^2 = \frac{+ 64a^2b^2}{-(1 + a^2)(1 + 2a - 6a^2 - 2a^3 + a^4) - 8a^2b^2} \qquad (2),$$

$$C_1^2 = \frac{16a^4(1 + b)^2}{(1 - a)^2(a - b)^6} \qquad (3),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} - \lambda\right)^2 = \frac{64a^4(-a + a^2 + a^3)}{(1 - a^2)^5(1 + 4a - a^2)} \qquad (4),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} + \lambda\right)^2 = \frac{(1 + a^2)^2(1 + 2a - 6a^2 - 2a^3 + a^4)^2}{(1 - a^2)^5(1 + 4a - a^2)} \qquad (5),$$

Now, as in the 'Archiv,' III., 1, p. 75, with the b there replaced by $-\frac{1 + a}{1 - a}$, and $\sqrt[4]{B}$ by $\frac{2b}{1 - a}$,

$$\mathrm{I}(v) = \int_x^0 \frac{-\dfrac{\mathrm{P}(v)}{\mathrm{M}}(\mathrm{D}-x^2) + \dfrac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\mathrm{D}-x^2} \frac{dx}{\sqrt{\mathrm{X}_1\mathrm{X}_2}},$$

$$= \tfrac{1}{5}\cos^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1 x + \mathrm{R}_2 x^2 + \mathrm{R}_3 x^3 + x^4}{(\mathrm{D}-x^2)}\sqrt{\tfrac{1}{2}\mathrm{X}_1}\,.$$

$$= \tfrac{1}{5}\sin^{-1}\frac{\mathrm{R}_0 - \mathrm{R}_1 x + \mathrm{R}_2 x^2 - \mathrm{R}_3 x^3 + x^4}{(\mathrm{D}-x^2)}\sqrt{\tfrac{1}{2}\mathrm{X}_2} \qquad (6),$$

$$\mathrm{X}_1, \quad \mathrm{X}_2 = 1 \pm \mathrm{C}x - x^2 \qquad (7),$$

$$\mathrm{D} = c^2 = \left(\frac{a-b}{a+b}\right)^2 = \frac{\mathrm{dn}^2 f\mathrm{K}'}{o^2}, \quad \mathrm{R}_0 = c^5 \qquad (8),$$

$$\frac{\mathrm{zn}\,\frac{1}{2}\mathrm{K}'}{o} = \frac{\mathrm{P}(5v)}{\sqrt{\mathrm{C}_0}} = \tfrac{1}{2}\left(\tfrac{1}{o} - o\right) \qquad (9),$$

$$\mathrm{P}(5v) = \tfrac{1}{2}\mathrm{C}_1 = \frac{2a^2(1+b)}{(1-a)(a-b)^3} \qquad (10),$$

$$\mathrm{C}_0 = \frac{a(1+3a-a^2-a^3-2ab^2)(a^2-b^2)^2}{4(1-a)^2 b^2} \qquad (11),$$

$$\frac{\mathrm{Q}(v)}{\mathrm{M}^3} = \frac{c^3}{\sqrt{\alpha(\alpha-m)}} \qquad (12),$$

$$\mathrm{R}_3\sqrt{\mathrm{C}_0} = -5\mathrm{P}(v) + \mathrm{P}(5v) = -1 - \frac{y^2}{y-x} = -\frac{2a}{a-b} - \frac{1+b}{1-a} \qquad (13),$$

while R_1 and R_2 can be calculated and the whole result verified by differentiating (6) and equating coefficients of powers of x.

A change of b into $-b$ will change v into $9v$.

The c employed for $\mu = 10$ is $\frac{1}{p'}$, so that, putting

$$\frac{1-c}{1+c} = \frac{p'-1}{p'+1} = a' \qquad (14),$$

we find

$$a' = -\frac{(a-b^2)^2}{a(1+a)^2} = \frac{1-2a-a^2+2b^2}{(1+a)^2} \qquad (15),$$

so that, with

$$u = \int_a^\infty \frac{da}{\sqrt{\mathrm{A}}}, \quad \mathrm{A} = 4b^4 \qquad (16),$$

$$a = a(u), \quad a' = a(u - \tfrac{1}{3}\omega) \qquad (17)$$

(KIEPERT, 'Math. Ann.,' 32, p. 107), and then, with $a = a_{20}$,

$$\eta_1 = a + 1 - \frac{1}{a},$$

$$\eta_2 = \frac{a^2 - 1}{a^2 - 4a - 1},$$

$$\eta_3 = -\frac{a^2 - 1}{4a}, \text{ \&c.} \qquad (18).$$

Then, in § 28,

$$\left(\frac{\Theta \frac{1}{10} \mathrm{K}'}{\Theta 0}\right)^{20} = c_1^{20} = \frac{m\alpha^7 (\alpha - m)^6 \gamma_9}{x^4 \gamma_{11}}, \text{ \&c.} \qquad (19).$$

43. $\mu = 28,\ v = \mathrm{K} + \frac{r\mathrm{K}'i}{14},\ r = 1, 3, 5, 9, 11, 13.$

$$\mathrm{D}_7 - c\mathrm{N}_7 = 0 \qquad (1).$$

$$\frac{a}{b} = \frac{\mathrm{N}_7 + \mathrm{D}_7}{\mathrm{N}_7 - \mathrm{D}_7} \qquad (2).$$

A change of b into $-b$ changes v into $13v$.

Now, with $1 - 2m = p$, $p\alpha = m(1 - m)\beta$, $8p\alpha = (1 - p^2)(\epsilon + 1)$,

$$\mathrm{D}_7 = m^2(1 - m)^2 \left\{ p\left(\frac{\epsilon + 1}{2}\right)^2 - \frac{3p - 1}{2}\,\frac{\epsilon + 1}{2} - \left(\frac{p - 1}{2}\right)^2 \right\}$$
$$= \tfrac{1}{4} m^2 (1 - m)^2 \{-p^2 + \epsilon(\epsilon - 1)p + \epsilon\} \qquad (3);$$

and changing p into $-p$,

$$\mathrm{N}_7 = \tfrac{1}{4} m^2 (1 - m)^2 \{-p^2 - \epsilon(\epsilon - 1)p + \epsilon\} \qquad (4).$$

$$\frac{\mathrm{N}_7 + \mathrm{D}_7}{\mathrm{N}_7 - \mathrm{D}_7} = \frac{\epsilon - p^2}{-\epsilon(\epsilon - 1)p} = \frac{a}{b} \qquad (5).$$

$$ap\epsilon(\epsilon - 1) + b(\epsilon - p^2) = 0 \qquad (6).$$

Substituting for p and ϵ in terms of a and b, and cancelling a factor $(a^2 - b^2)^2$, there results

$$ab^8 + (1 + a - a^2 - 3a^3)b^4 - a^3(1 + a - 2a^2 - a^3) = 0 \qquad (7),$$

a quadratic for b^4.

To obtain a rational expression for C, we put

$$a^3 - b^4 = a^3(1 - a)c^2 \qquad (8),$$

and now the relation (7) becomes

$$(a^2c^2 + a)^2 - (1 + a)^2 c^2 = 0 \qquad (9),$$

$$a^2c^2 + ac + a + c = 0 \qquad (10),$$

$$c = \frac{-1 - a + \sqrt{A}}{2a^2} \qquad (11),$$

$$A = (1 + a)^2 - 4a^3 = (1 - a)(1 + 3a + 4a^2) \qquad (12);$$

so that, with

$$a = a(u), \quad u = \int_a^x \frac{da}{\sqrt{A}} \qquad (13),$$

then

$$c = a(u - \tfrac{1}{3}\omega) \qquad (14)$$

(KIEPERT, 'Math. Ann.,' 32, p. 108).

Introducing these values into (35), § 41,

$$\tfrac{1}{4}\left(\frac{1}{\lambda} - \lambda\right)^2 = \frac{-128a^7(1 + a - 2a^2 - a^3)}{(1 + a)^7(1 - a)^3[7a(1 - a^2) + (2 - a + a^2)\sqrt{A}]} \qquad (15),$$

to be compared with the expression for ξ_2 in $\mu = 14$, L.M.S., 27, p. 440, putting

$$c = \frac{1 - a}{2a}, \quad 1 + 2c = \frac{1}{a} \qquad (16).$$

Writing q or p' for p_{14},

$$\begin{aligned} q = p' &= \frac{p^2 - \epsilon}{(\epsilon + 1)p^2} \\ &= -\frac{-a^3 + (-a^2 + a^3 + a^4)b^2 + (1 + a - a^2)b^4 - ab^6}{(1 + a)(a - b^2)(a^2 - b^4)} \qquad (17), \end{aligned}$$

and after reduction

$$\begin{aligned} (1 + a)^2q^2 - a(1 + 3a - \sqrt{A})q \\ + a^2(1 - a) - a\sqrt{A} = 0 \qquad (18), \end{aligned}$$

or

$$q^2 - a(t + 1)q + at = 0, \quad t = a(u - \tfrac{2}{3}\omega) \qquad (19),$$

to be interpreted as an elliptic function relation.

Rationalising again

$$\begin{aligned} (1 + a)^2q^4 - 2a(1 + 3a)q^3 + 2a^2(1 + a)q^2 \\ + 2a^2(1 - a)q - a^2(1 - a^2) = 0 \qquad (20), \end{aligned}$$

or arranged in powers of a,

$$\begin{aligned} a^4 + 2q(q - 1)a^3 + (q^4 \pm 6q^3 + 2q^2 \mp 2q - 1)a^2 \\ + 2q^3(q - 1)a + q^4 = 0 \qquad (21), \end{aligned}$$

$$a^2 + q(q - 1)a + q^2 = a\sqrt{Q} \qquad (22),$$

$$Q = 4q^3 + (q - 1)^2 \qquad (23),$$

44. $$\mu = 36, \quad v = \mathrm{K} + \frac{2r+1}{18}\mathrm{K}'i.$$

The relation
$$\{s(\omega) - s(9v)\}^2 = (s_1 - s_3)(s_2 - s_3) \tag{1}$$
leads to
$$\frac{\mathrm{N}_9}{\mathrm{D}_9} = \frac{1+b}{1-b} \quad (2),$$

$$b = \frac{\mathrm{N}_9 + \mathrm{D}_9}{\mathrm{N}_9 - \mathrm{D}_9} = \frac{\epsilon^2[(\epsilon - 2)p^2 + \epsilon]}{p[p^2 - \epsilon(\epsilon^2 - \epsilon + 1)]} \quad (3),$$

and putting
$$p^2 - 1 = q, \quad \frac{\epsilon - 1}{q} = r, \quad 2r - 1 = s, \quad \frac{\epsilon - 1}{s} = t, \quad t + 1 = u \quad (4),$$

$$\frac{2\epsilon^2}{bp} + \epsilon + 1 - \epsilon\frac{\epsilon - 1}{u} = 0 \quad (5),$$

and thereby the superfluous factor $a^2 - b^2$ is cancelled, and finally

$$a^3b^{12} + (1 + 3a + 0 - 6a^3 + 0 + 0 - a^6)\,b^8$$
$$- 3(1 + 2a - 2a^2 - 2a^3)\,a^3b^4 + (3 + 0 - 3a^2 - a^3)\,a^6 = 0 \quad (6),$$

a cubic equation for b^4.

Putting as before
$$a^3 - b^4 = \frac{a^3(1-a)}{(c+1)^2} \quad (7),$$
this relation (63) becomes

$$[c^3 + 2(1 - a^2)c]^2 - (1-a)^2[(2+a)c^2 + (1+a)^2]^2 = 0 \quad (8),$$

$$c^3 + (1-a)(2+a)c^2 + 2(1-a^2)c + (1-a)(1+a)^2 = 0 \quad (9),$$

and putting $c = (1 + a)q$,

$$(1+a)q^3 - (1-a)(2+a)q^2 + 2(1-a)q + 1 - a = 0 \quad (10),$$

and arranged as a quadratic in a,

$$q^2a^2 - (q^3 - q^2 - 2q - 1)\,a - (q+1)(q^2 + q + 1) = 0 \quad (11),$$

$$a = \frac{q^3 - q^2 - 2q - 1 + \sqrt{\mathrm{Q}}}{2q^2} \quad (12),$$

$$\mathrm{Q} = q^6 + 2q^5 + 5q^4 + 10q^3 + 10q^2 + 4q + 1 \quad (13),$$
as in $\mu = 18$, § 32; and
$$c = \frac{q^3 + q^2 - 2q - 1 \pm \sqrt{\mathrm{Q}}}{2q} \quad (14).$$

45. $\mu = 44, \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \dfrac{2r+1}{22}.$

Here the relation to be reduced becomes

$$\frac{b}{a} = \frac{\mathrm{N}_{11} + \mathrm{D}_{11}}{\mathrm{N}_{11} - \mathrm{D}_{11}} = \frac{\epsilon\,[(\epsilon - 2)\,p^2 + \epsilon]\,[(\epsilon^2 - \epsilon - 1)\,p^2 + \epsilon]}{p\,[p^2 - \epsilon^2]\,[p^2 - \epsilon^3 + \epsilon^2 - \epsilon]} \qquad (1).$$

The algebraical work has been carried out by Mr. R. H. MACMAHON, and he reduces the relation to a quintic in b^4,

$$\mathrm{A}_0 + \mathrm{A}_1 b^4 + \mathrm{A}_2 b^8 + \mathrm{A}_3 b^{12} + \mathrm{A}_4 b^{16} + \mathrm{A}_5 b^{20} = 0 \qquad (2),$$

$$\begin{array}{lllllllllll}
\mathrm{A}_0 = a^{10} & (1 - & 2a - & 5a^2 + & 2a^3 + & 4a^4 + & a^5), & & & & \\
\mathrm{A}_1 = a^7\,(- & 1 + & 2 & + 15 & + 8 & - 19 & - 10a^5), & & & & \\
\mathrm{A}_2 = a^6\,(- & 15 - & 27 & + 14 & + 45 & + 0 & - 11 + & 0 + & 3 + & a^8), & \\
\mathrm{A}_3 = a^3 & (6 + & 22 & + 5 & - 40 & - 23 & + 26 + & 5 - & 8 - & 3a^8), & \\
\mathrm{A}_4 = & - & 1 - & 5a - & 5a^2 + 9 & + 13 & - 5 - & 10 + & 6 + & 3a^8, & \\
\mathrm{A}_5 = - a^5 & & & & & & & & & & (3).
\end{array}$$

As before, putting

$$a^3 - b^4 = \frac{a^3\,(1 - a)}{(v + 1)^2}, \text{ and } v = (1 + a)\,c \qquad (4),$$

the relation can be halved in degree, and becomes

$$\begin{aligned} c^5 a^4 + c^2\,(c+1)\,(2c^2 + 2c + 1)\,a^3 + (c^5 + 3c^4 + 6c^3 + 8c^2 + 4c + 1)\,a^2 \\ - c^2\,(2c^2 + 5c + 3)\,a - (c + 1)^4 = 0 \qquad (5), \end{aligned}$$

a quartic in a of similar structure to the one in $\mu = 22$, § 33 (3), and capable of resolution in a similar manner into

$$[2c^3a^2 + (c+1)\,(2c^2 + 2c + 1)\,a - (c+1)]^2 = \mathrm{C}\,[(c - 1)\,a + (c + 1)]^2 \qquad (6),$$

$$\mathrm{C} = 4c\,(c + 1)^2 + 1 \qquad (7).$$

Put

$$\frac{2c^3a^2 + (c+1)\,(2c^2 + 2c + 1)\,a - (c+1)}{(c - 1\ \ a + (c + 1)} = 2c^2x - 2c - 1 \qquad (8),$$

and then

$$x = \frac{2c + 1 + \sqrt{\mathrm{C}}}{2c^2} = c\,(u - \tfrac{4}{5}\,\omega) \qquad (9),$$

if

$$c = c\,(u), \quad u = \int_c^\infty \frac{dc}{\sqrt{\mathrm{C}}} \qquad (10);$$

and

$$cx = \frac{c^2a^2 + (c^2 + 3c + 1)\,a + c + 1}{(c - 1)\,a + (c + 1)} \qquad (11),$$

a relation which requires interpretation, connecting c, x, and a, elliptic functions of u.

It is now possible theoretically to determine the coefficients in

$$\mathrm{I}(v) = \int_x^0 \frac{-\frac{\mathrm{P}(v)}{\mathrm{M}}(\delta - x^2) + \frac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\delta - x^2} \frac{dx}{\sqrt{\mathrm{X}_1\mathrm{X}_2}}$$

$$= \frac{1}{11} \cos^{-1} \frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_9 x^9 - x^{10}}{(\delta - x^2)^{\frac{11}{2}}} \sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{11} \sin^{-1} \frac{\mathrm{R}_0 - \mathrm{R}_1 x + \ldots - \mathrm{R}_9 x^9 - x^{10}}{(\delta - x^2)^{\frac{11}{2}}} \sqrt{\tfrac{1}{2}\mathrm{X}_2} \qquad (12),$$

and to construct an algebraical herpolhode and associated top motion complete in 44 branches.

This is as far as we can go at present, as the next cases of $\mu = 52$ and 60 must await the solution of $\mu = 26$ and 30, not yet accomplished.

46. II. $\quad \frac{1}{4}\mu = 2n, \quad \mu = 8n, \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \frac{2r+1}{4n}.$

$$\mathrm{X}_1 = \mathrm{C}_0 + \mathrm{C}_1 x + x^2 = (x_1 + x)(x_2 + x) \qquad (1),$$

$$\mathrm{X}_2 = \mathrm{C}_0 - \mathrm{C}_1 x + x^2 = (x_1 - x)(x_2 - x) \qquad (2),$$

$$\mathrm{C}_0 = s(\omega_3) - s(2nv) = \sqrt{(s_1 - s_3 \,.\, s_2 - s_3)} \qquad (3),$$

$$\mathrm{C}_1 = \sqrt{(s_1 - s_3)} + \sqrt{(s_2 - s_3)} = 2\mathrm{P}(2nv) \qquad (4),$$

$$\mathrm{C}_0\mathrm{C}_1 = \tfrac{1}{2}i\wp'(2nv) = \mathrm{Q}(2nv) \qquad (5),$$

$$\left(\tfrac{1}{o} + o\right)^2 = \frac{\mathrm{C}_1{}^2}{\mathrm{C}_0} = \mathrm{C}^2 \qquad (6),$$

and normalised by $\mathrm{M} = \sqrt{\mathrm{C}_0}$,

$$\mathrm{X}_1,\ \mathrm{X}_2 = 1 \pm \mathrm{C}x + x^2, \quad x_1 = \tfrac{1}{o}, \quad x_2 = o \qquad (7),$$

$$x = o\,\mathrm{sn}(\mathrm{K} - mt), \quad \frac{\mathrm{dn}^2 f\mathrm{K}'}{o^2} = \mathrm{D} \qquad (8),$$

$$\frac{\mathrm{zn} f\mathrm{K}'}{o} = \frac{\mathrm{P}(v)}{\mathrm{M}} \qquad (9),$$

and the associated integral

$$\mathrm{I}(v) = \int_x^0 \frac{-\frac{\mathrm{P}(v)}{\mathrm{M}}(\mathrm{D} - x^2) + \frac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\mathrm{D} - x^2} \frac{dx}{\sqrt{(\mathrm{X}_1\mathrm{X}_2)}}$$

$$= \frac{1}{2n} \cos^{-1} \frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n-2} x^{2n-2} + \mathrm{R}_{2n-1} x^{2n-1}}{(\mathrm{D} - x^2)^n} \sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{2n} \sin^{-1} \frac{\mathrm{R}_0 - \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n-2} x^{2n-2} - \mathrm{R}_{2n-1} x^{2n-1}}{(\mathrm{D} - x^2)^n} \sqrt{\tfrac{1}{2}\mathrm{X}_2} \qquad (10)$$

$$R_0{}^2 = D^{2n}, \quad R_{2n-2} = -[2nP(v) + P(2nv)]\,R_{2n-1}, \quad R_{2n-1} = (-1)^n \tag{11},$$

$$\frac{Q(v)}{M^3} = \left(\frac{1}{o^2} - o^2\right) \operatorname{sn} f K' \operatorname{cn} f K' \operatorname{dn} f K' \tag{12}.$$

In this case II. put

$$(\alpha - m)(\alpha - m + 1) = \alpha^2 a^4 \tag{13},$$

then

$$C_0{}^2 = \frac{m^4 \alpha^4 a^4}{(\alpha - m)^4}, \quad C_0 = a^2 [s(\omega) - s(2v)] \tag{14},$$

$$\delta = s(v) - s(\omega_3) = -\frac{m^2 \alpha}{\alpha - m}\ \text{(positive)} \tag{15}.$$

Now, with

$$1 - 2m = p, \quad (1 - 2m)\alpha = m(1 - m)\beta = m(1 - m)\frac{\epsilon + 1}{2} \tag{16},$$

$$\frac{s(\omega) - s(4v)}{s(\omega) - s(2v)} = \left(\frac{\beta - 1}{\beta}\right)^2 = \left(\frac{\epsilon - 1}{\epsilon + 1}\right)^2 = b^2,\ \text{suppose} \tag{17};$$

$$\epsilon = \frac{1 - b}{1 + b}, \quad \beta = \frac{1}{1 + b} \tag{18};$$

and the relation (13) becomes

$$1 - a^4 + \frac{p}{\alpha} - \frac{1 - p^2}{4\alpha^2} = 0 \tag{19},$$

$$1 - a^4 + \frac{4p^2(1 + b)}{1 - p^2} - \frac{4p^2(1 + b)^2}{1 - p^2} = 0 \tag{20},$$

$$p^2 = \frac{1 - a^4}{(1 + 2b)^2 - a^4}, \quad m - m^2 = \frac{1 - p^2}{4} = \frac{b(1 + b)}{(1 + 2b)^2 - a^4} \tag{21},$$

$$p\alpha = \tfrac{1}{4}(1 - p^2)\beta = \frac{b}{(1 + 2b)^2 - a^4} \tag{22},$$

$$\frac{p}{\alpha} = \frac{p^2}{p\alpha} = \frac{1 - a^4}{b} \tag{23},$$

$$\frac{1}{\alpha^2} = \frac{p^2}{p^2\alpha^2} = \frac{(1 - a^4)[(1 + 2b)^2 - a^4]}{b^2} \tag{24},$$

$$\begin{aligned}
\left(\tfrac{1}{o} + o\right)^2 = C'^2 &= \frac{C_1{}^2}{C_0} \\
&= 2 + \frac{1 - 4(1 - 2m)\alpha - 8\alpha^2}{4\alpha^2 a^2} \\
&= 2 + \frac{1}{4\alpha^2 a^2} - \frac{p}{\alpha a^2} - \frac{2}{a^2} \\
&= 2 + \frac{(1 - a^4)[(1 + 2b)^2 - a^4]}{4a^2 b^2} - \frac{1 - a^4}{a^2 b} - \frac{2}{a^2} \\
&= \frac{(1 - a^2)^2[(1 + a^2)^2 - 4b^2]}{4a^2 b^2}
\end{aligned} \tag{25},$$

$$\left(\tfrac{1}{o} - o\right)^2 = \frac{(1 + a^2)^2 [(1 - a^2)^2 - 4b^2]}{4a^2 b^2} \tag{26},$$

$$1 - 8(1 - 2m)\,\alpha = \frac{(1 - 2b)^2 - a^4}{(1 + 2b)^2 - a^4} \tag{27}.$$

We now find, as in § 41, writing f for $f\text{K}'$,

$$\frac{\text{dn}^2 f}{o^2} = \frac{\sigma - s_3}{\text{C}_0} = \frac{1}{a^2}\left(\frac{m}{\alpha} - 1\right)$$

$$= \frac{\sqrt{1 + a^2}\sqrt{2b + 1 - a^2} - \sqrt{1 - a^2}\sqrt{2b + 1 + a^2}}{\sqrt{1 + a^2}\sqrt{2b + 1 - a^2} + \sqrt{1 - a^2}\sqrt{2b + 1 + a^2}} \tag{28},$$

$$\frac{\text{dn} f}{o} = \frac{\sqrt{1 + a^2}\sqrt{2b + 1 - a^2} - \sqrt{1 - a^2}\sqrt{2b + 1 + a^2}}{2a\sqrt{b}} \tag{29},$$

and then, from the relation

$$\frac{\text{dn}\,(2n + 1)\,f}{\text{dn} f} = \frac{\text{N}_{2n+1}}{\text{D}_{2n+1}} \tag{30},$$

writing c for a^2, and putting

$$(1 + c)(2b + 1 - c) = \text{B}_1, \quad (1 - c)(2b + 1 + c) = \text{B}_2 \tag{31},$$

$$\frac{\text{dn}3f}{o} = \frac{(b + c)\sqrt{\text{B}_1} - (b - c)\sqrt{\text{B}_2}}{2(b + 1)\sqrt{bc}} \tag{32},$$

$$\frac{\text{dn}5f}{o} = \frac{(b^3 - b^2c - bc^2 + c)\sqrt{\text{B}_1} - (b^3 + b^2c - bc^2 - c)\sqrt{\text{B}_2}}{2(b^3 - b^2 - b + c^2)\sqrt{bc}} \tag{33},$$

$$\frac{\text{dn}7f}{o} = \frac{\text{E}_7\sqrt{\text{B}_1} - \text{F}_7\sqrt{\text{B}_2}}{2\{(b^2 - 1)(b^4 - c^2) + b(b^2 - c^2)(2b^2 - 1 - c^2)\}\sqrt{bc}} \tag{34},$$

$$\text{E}_7 = b^6 + 2b^5c - b^4c^2 - c(3 + c^2)b^3 - b^2c^2 + c(1 + c^2)b + c^4$$
$$= (b^2 - c^2)^2(b^2 + c^2) + bc(b^2 - 1)(2b^2 - 1 - c^2),$$

and F_7 is obtained from E_7 by changing c into $-c$.

47. $\mu = 16, \quad v = \text{K} + f\text{K}'i, \quad f = \frac{1,\,3,\,5,\,7}{8},$

and now the condition

$$s(\omega) - s(4v) = \sqrt{(s_1 - s_3,\, s_2 - s_3)} = \text{C}_0 \tag{1}$$

reduces to

$$a = b \tag{2},$$

$$\frac{1}{o} + o = C = \tfrac{1}{2}\left(b - \frac{1}{b}\right)^2 \qquad (3),$$

$$\frac{1}{o} - o = \frac{(b^2 + 1)\sqrt{(b^4 - 6b^2 + 1)}}{2b^2} \qquad (4),$$

$$D = \frac{\operatorname{dn}^2 f}{o^2} = \frac{[\sqrt{(b^2 + 1)}\sqrt{(b^2 - 2b - 1)} - (b + 1)\sqrt{(b^2 - 1)}]^2}{4b^3} \qquad (5),$$

$$\begin{aligned} I(v) &= \tfrac{1}{4}\cos^{-1}\frac{R_0 + R_1 x + R_2 x^2 + x^3}{(D - x^2)^2}\sqrt{\tfrac{1}{2}X_1}, \\ &= \tfrac{1}{4}\sin^{-1}\frac{R_0 - R_1 x + R_2 x^2 - x^3}{(D - x^2)^2}\sqrt{\tfrac{1}{2}X_2} \end{aligned} \qquad (6),$$

$$X_1,\ X_2 = 1 \pm \tfrac{1}{2}\left(b - \frac{1}{b}\right)^2 x + x^2 \qquad (7),$$

$$R_0 = D^2 \qquad (8),$$

$$R_1 = \frac{\left\{\begin{array}{c} -b^8 + 0 - 4 - 4 - 2 + 4 + 6 + 4 + 1 - (b + 1)^2 \\ (b^3 - b^2 - b - 1)\sqrt{(b^4 - 1)(b^2 - 2b - 1)} \end{array}\right\}}{2b^4} \qquad (9),$$

$$R_2 = -\frac{2b + 1 + (b + 1)\sqrt{(b^4 - 1)(b^2 - 2b - 1)}}{b^2} \qquad (10),$$

$$\frac{P(v)}{M} = \frac{\operatorname{zn} f\, K'}{o} = \frac{4(b + 1)\sqrt{(b^4 - 1)(b^2 - 2b - 1)} - (b^2 + 2b + 3)(b^2 - 2b - 1)}{16b^2} \qquad (11),$$

$$\frac{Q(v)}{M^3} = \frac{(b + 1)\sqrt{(b^4 - 1)(b^2 - 2b - 1)}}{2b^2}D \qquad (12).$$

Bisection formulas for $\mu = 8$ in the region $v = \frac{1}{4}K'i$ will lead to the same results, with $a_8 = a$,

$$\sqrt{a} = \frac{\sqrt{b^2 + 2b - 1} + \sqrt{b^2 - 2b - 1}}{2b} \qquad (13),$$

$$\frac{1}{\sqrt{a}} = \frac{\sqrt{b^2 + 2b - 1} - \sqrt{b^2 - 2b - 1}}{2} \qquad (14),$$

$$b^2 = \frac{1 + a}{a - a^2} \qquad (15),$$

having the substitution

$$\left(a,\ -\frac{1}{a}\right), \quad \left(b,\ -\frac{1}{b}\right) \qquad (16).$$

Hence the following table of section-values for $\mu = 16$, holding in the region $\infty > b > \sqrt{2} + 1$; tested numerically by $b = \left(\frac{\sqrt{5} + 1}{2}\right)^2$, $o = \frac{1}{2}$, $\kappa = \frac{1}{4}$, as on p. 278, region C:—

$\frac{1}{\sqrt{2}}\left(b+\frac{1}{b}\right)$	$\frac{1}{\sqrt{o}}+\sqrt{o}$
$\frac{1}{\sqrt{2}}\sqrt{\left(b^2-6+\frac{1}{b^2}\right)}$	$\frac{1}{\sqrt{o}}-\sqrt{o}$
$\frac{1}{2}\left(b-\frac{1}{b}\right)^2$	$\frac{1}{o}+o$
$\frac{b^2+1\pm\sqrt{(b^4-6b^2+1)}}{2\sqrt{2b}}$	$\frac{1}{\sqrt{o}},\ \sqrt{o}$
$\frac{\sqrt{(b^2+2\sqrt{2b}+1)}\pm\sqrt{(b^2-2\sqrt{2b}+1)}}{2\sqrt[4]{2}\sqrt{b}}$	$\frac{1}{\sqrt[4]{o}},\ \sqrt[4]{o}$
$\frac{(b+1)\sqrt{b^2-1}+\sqrt{(b^2+1)(b^2-2b-1)}}{2b^3}$	$\frac{\operatorname{dn}\frac{3}{8}\mathrm{K}'}{o}$
$\frac{(b-1)\sqrt{b^2-1}+\sqrt{(b^2+1)(b^2-2b-1)}}{2b^4}$	$\frac{\operatorname{dn}\frac{1}{8}\mathrm{K}'}{o}$
$\frac{(b+1)\sqrt{b^2+2b-1}-(b-1)\sqrt{b^2-2b-1}}{(b+1)\sqrt{b^2+2b-1}+(b-1)\sqrt{b^2-2b-1}}$	$\operatorname{dn}\frac{1}{4}\mathrm{K}'$
$\frac{(b-1)\sqrt{b^2+2b-1}+(b+1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{sn}\frac{1}{4}\mathrm{K}'$
$\frac{(b+1)\sqrt{b^2+2b-1}-(b-1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{cn}\frac{1}{4}\mathrm{K}'$
$\frac{(b-1)\sqrt{b^2+2b-1}-(b+1)\sqrt{b^2-2b-1}}{(b-1)\sqrt{b^2+2b-1}+(b+1)\sqrt{b^2-2b-1}}$	$\operatorname{dn}\frac{3}{4}\mathrm{K}'$
$\frac{(b-1)\sqrt{b^2+2b-1}-(b+1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{cn}\frac{3}{4}\mathrm{K}'$
$\frac{(b+1)\sqrt{b^2+2b-1}+(b-1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{sn}\frac{3}{4}\mathrm{K}'$
$\frac{b^2+1-\sqrt{b^4-6b^2+1}}{b^2+1+\sqrt{b^4-6b^2+1}}$	$\operatorname{dn}\frac{1}{2}\mathrm{K}'$
$\frac{b^2+1+\sqrt{b^4-6b^2+1}}{2(b^2-1)}$	$\operatorname{sn}\frac{1}{2}\mathrm{K}'$
$\frac{b^2+1-\sqrt{b^4-6b^2+1}}{2(b^2-1)}$	$\operatorname{cn}\frac{1}{2}\mathrm{K}'$

$\frac{\sqrt{b^2-2b-1}}{16b^2}\left[4(b-1)\sqrt{b^4-1}+(3b^2-2b+1)\sqrt{b^2-2b-1}\right]$	$\frac{\mathrm{zn}\,\frac{1}{8}\,\mathrm{K}'}{o}$
$\frac{\sqrt{b^2-2b-1}}{16b^2}\left[4(b-1)\sqrt{b^4-1}-(3b^2-2b+1)\sqrt{b^2-2b-1}\right]$	$\frac{\mathrm{zn}\,\frac{7}{8}\,\mathrm{K}'}{o}$
$\frac{\sqrt{b^2-2b-1}}{16b^2}\left[4(b+1)\sqrt{b^4-1}+(b^2+2b+3)\sqrt{b^2-2b-1}\right]$	$\frac{\mathrm{zn}\,\frac{3}{8}\,\mathrm{K}'}{o}$
$\frac{\sqrt{b^2-2b-1}}{16b^2}\left[4(b+1)\sqrt{b^4-1}-(b^2+2b+3)\sqrt{b^2-2b-1}\right]$	$\frac{\mathrm{zn}\,\frac{5}{8}\,\mathrm{K}'}{o}$
$c_1{}^{32} = \dfrac{b^{25}\left[\dfrac{(b+1)\sqrt{b^2-1}+\sqrt{(b^2+1)(b^2-2b-1)}}{2b^{\frac{1}{2}}}\right]^{16}}{(b-1)^7(b+1)^{15}(b^2-2b-1)^4(b^2+1)^6}$	$\left(\frac{\Theta\,\frac{1}{8}\,\mathrm{K}'}{\Theta 0}\right)^{32}$
$c_2{}^8 = \dfrac{b^9}{(b^2-1)^3(b^2+1)^2}$	$\left(\frac{\mathrm{H}\,\frac{1}{4}\,\mathrm{K}'}{\mathrm{HK}'}\right)^8$

48. $\mu = 24, \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \frac{1,\,5,\,7,\,11}{12}.$

$$s(\omega) - s(6v) = \mathrm{C}_0 = \frac{m^2\alpha^2a^2}{(\alpha - m)^2} \qquad (1),$$

and this leads to

$$(1 - 2m + 2m^2)\,\beta - (1 - 3m + 3m^2)$$
$$+ (2\beta - 1)(m - m^2)\,a = 0 \qquad (2),$$

$$m(1 - m)[3 - a - 2(1 - a)\,\beta] + \beta - 1 = 0 \qquad (3),$$

$$(1 + b)[a(1 - b) + 1 + 3b] - (1 + 2b)^2 + a^4 = 0 \qquad (4),$$

$$a(1 - b^2) - b^2 + a^4 = 0 \qquad (5),$$

$$b^2 = \frac{a + a^4}{1 + a} = a - a^2 + a^3 \qquad (6),$$

$$\left(\tfrac{1}{o} + o\right)^2 = \frac{(1 - a)^6(1 + a)^2}{4a^2(a - a^2 + a^3)} \qquad (7),$$

$$\left(\tfrac{1}{o} - o\right)^2 = \frac{(1 + a^2)^3(1 - 4a + a^2)}{4a^2(a - a^2 + a^3)} \qquad (8),$$

derivable from the results for $\mu = 12$ by putting $a_{12} = c$,

$$c + \frac{1}{c} = \frac{(a - 1)^2}{a} \qquad (9),$$

and now

$$\underset{o}{\mathrm{dn}}\tfrac{1}{12}K' = \frac{\sqrt{1+a^2}\sqrt{(1-a^2+2\sqrt{a-a^2+a^3})}-\sqrt{1-a^2}\sqrt{(1+a^2+2\sqrt{a-a^2+a^3})}}{2a\sqrt[4]{a-a^2+a^3}}, \text{ \&c.} \tag{10}$$

In KIEPERT'S notation ('Math. Ann.,' 32, p. 116),

$$\sqrt{\xi} = \frac{1-a^2+\sqrt{a-a^2+a^3}}{1+a^2}, \quad \xi_1 = \frac{1-4a+a^2}{1+a^2}, \quad \xi_2 = \frac{a}{1-a+a^2}, \text{ \&c.} \tag{11}$$

Treating $\mu = 24$ by the trisection method of § 14 applied to $\mu = 8$, we put

$$x = z(1-2z), \quad y = \frac{z(1-2z)}{1-z}, \quad y+1 = \frac{1-2z^2}{1-z} \tag{12;}$$

so that, putting $t = z^2p$ in the equation (8), § 14, it may be written

$$(\tfrac{1}{o}+o)^2 = \frac{(1-2z)^4}{4z^2(1-z)^2} = -\frac{p^3(3p+4)}{4(p+1)^3} \tag{13}$$

To agree with the notation in KLEIN-FRICKE, 'Modul-Functionen,' 1, p. 688, put

$$p+1 = \tfrac{1}{3}y \tag{14,}$$

$$\lambda = \tfrac{1}{4}(\tfrac{1}{o}+o)^2 = \frac{(3-y)^3(1+y)}{16y^3}, \quad \lambda - 1 = \frac{(3+y)^3(1-y)}{16y^3} \tag{15,}$$

$$\lambda(\lambda-1) = \frac{(9-y^2)^3(1-y^2)}{256y^6} = \frac{(8+x^3)^3x^3}{256(1-x^3)^3} \tag{16,}$$

$$x^3+y^2 = 1 \tag{17:}$$

and denoting the *tetrahedron-irrationality* by ξ,

$$\xi = \frac{x^3-4}{-3x^2}, \quad \sqrt{(\xi^3-1)} = \frac{(y^2-9)y}{3^{\frac{3}{2}}(y^2-1)} \tag{18}$$

$$x = -\tau_{18}-2, \quad y = \tau_{12}-3 \tag{19}$$

$$2\tau_6+8 = (\tau_{18}+2)^3 = -x^3, \quad 2\tau_6+9 = (\tau_{12}-3)^2 = y^2 \tag{20;}$$

and GIERSTER'S τ_{12} ('Math. Ann.,' 14) is given by

$$\tau_{12} = \frac{36}{y^2-9} = \frac{36}{-x^3-8} \tag{21}$$

So also $\mu = 48$ can be discussed by a trisection of $\mu = 16$, by putting

$$(\tfrac{1}{o}+o)^2 = \tfrac{1}{4}\left(\frac{1}{a}-a\right)^4 = \frac{(b-1)^3(b+3)}{4b} \tag{22}$$

49. $\mu = 32,\ v = \mathrm{K} + f\mathrm{K}'i,\ f = \frac{1,\,3,\,5,\,7,\,9,\,11,\,13,\,15}{16}$.

$$[s(\omega) - s(8v)]^2 = \frac{m^4\alpha^4 a^4}{(\alpha - m)^4} \tag{1},$$

and this leads to

$$\beta^3 - (1 + 2m - 2m^2)\beta^2 + 4(m - m^2)\beta - (m - m^2) \\ - \beta(\beta - 1)(\beta - 1 + 2m - 2m^2)a = 0 \tag{2},$$

$$m - m^2 = \frac{\beta(\beta - 1)[\beta(1 - a) + a]}{2\beta^2(1 + a) - 2\beta(2 + a) + 1} \\ = \frac{b(1 + ab)}{(1 + b)(1 + 2b - b^2 + 2ab)} \tag{3},$$

so that

$$(1 + ab)\{(1 + 2b)^2 - a^4\} \\ - (1 + b)^2(1 + 2b - b^2 + 2ab) = 0 \tag{4},$$

$$b^4 + 2ab^3 - a(a^4 + 1)b - a^4 = 0 \tag{5},$$

a C_6 in (a, b).

Put $b = ac$,

$$a^4c - (c^4 + 2c^3 + 0 + 0 - 1)a^2 + c = 0 \tag{6},$$

a quadratic in a^2,

$$a^2 + \frac{1}{a^2} = \frac{c^4 + 2c^3 + 0 + 0 - 1}{c} \tag{7},$$

$$\left(a + \frac{1}{a}\right)^2 = \frac{(c^2 + 1)(c^2 + 2c - 1)}{c} \tag{8},$$

$$\left(a - \frac{1}{a}\right)^2 = \frac{(c + 1)^3(c - 1)}{c} \tag{9},$$

$$a = \frac{\sqrt{c^2 + 1}\,.\,\sqrt{c^2 + 2c - 1} + (c + 1)\sqrt{c^2 - 1}}{2\sqrt{c}} \tag{10},$$

$$b = \frac{\sqrt{c^2 + 1}\,.\,\sqrt{c^2 + 2c - 1} + (c + 1)\sqrt{c^2 - 1}}{2}\sqrt{c} \tag{11},$$

$$\mathrm{C}^2 = \tfrac{1}{4}\left(\frac{1}{a} - a\right)^2\left[\frac{1}{c^2}\left(\frac{1}{a} + a\right)^2 - 4\right] \\ = \frac{(c + 1)^3(c - 1)}{4c}\left[\frac{(c^2 + 1)(c^2 + 2c - 1)}{c^3} - 4\right] \\ = \frac{(c^2 - 1)^4}{4c^4} \tag{12},$$

$$\mathrm{C} = \frac{1}{o} + o = \tfrac{1}{2}\left(c - \frac{1}{c}\right)^2 \tag{13},$$

$$\frac{1}{\sqrt{o}} + \sqrt{o} = \frac{1}{\sqrt{2}}\left(c + \frac{1}{c}\right) \qquad (14),$$

as in $\mu = 16$, so that the bisection formulas can be carried one stage further.

We find now

$$\underset{o}{\text{dn}\,\tfrac{1}{16}\text{K}'} = \frac{\sqrt{\left[1 + \dfrac{2c^{\frac{3}{2}}}{\sqrt{(c^2+1)(c^2+2c-1)}}\right]} - \sqrt{\left[1 - \dfrac{2c^{\frac{3}{2}}}{(c+1)\sqrt{c^2-1}}\right]}}{\sqrt{\left[1 + \qquad ,, \qquad\right]} + \sqrt{\left[1 - \qquad ,, \qquad\right]}} \qquad (15).$$

50. $\mu = 40, \quad v = \text{K} + f\text{K}'i, \quad f = \dfrac{2r+1}{20}, \quad = \dfrac{1, 3, 7, 9, 11, 13, 17, 19}{20}.$

The relation

$$[s(\omega) - s(10v)]^2 = (s_1 - s_3)(s_2 - s_3) \qquad (1)$$

leads to

$$\text{N}_{10} + a\text{D}_{10} = 0 \qquad (2),$$

or putting $m - m^2 = n$,

$$(1-4n)\beta^4 - 2(2-7n)\beta^3 + 2(3-11n+2n^2)\beta^2 - (4-17n+10n^2)\beta + 1 - 5n + 5n^2$$
$$+ a(\beta^2 - \beta + n)[(1-4n)\beta^2 - (1-bn)\beta - n] = 0 \qquad (3).$$

With $\beta = \dfrac{1}{b+1}$, and arranged in powers of n,

$$(b+1)^2[5b^2 - 1 - a(b^2 - 4b - 1)]n^2$$
$$- [5b^3 + 3b^2 + b - 1 + 2a(2b+1)]bn$$
$$+ b^2(b^2 + a) = 0 \qquad (4),$$

and with

$$\frac{b}{n} = \frac{(1+2b)^2 - a^4}{(1+b)} \qquad (5)$$

this becomes

$$(b+1)^4[5b^2 - 1 - a(b^2 - 4b - 1)]$$
$$- (b+1)[5b^3 + 3b^2 + b - 1 + 2a(3b+1)][(2b+1)^2 - a^4]$$
$$+ (b^2 + a)[(2b+1)^2 - a^4]^2 = 0 \qquad (6).$$

A factor $1 - a$ may be cancelled and

$$b^6 + 0\,.\,b^5 + \text{A}_2b^4 + \text{A}_3b^3 + \text{A}_4b^2 + 0b$$
$$- a^4(1 + a + a^2 + a^3 + a^4) = 0 \qquad (7).$$

$$\text{A}_2 = 3a(1 + a + a^2), \quad \text{A}_3 = 0, \quad \text{A}_4 = -a(1 + a + a^2 - a^3 + a^4 + a^5 + a^6) \qquad (8).$$

Put $b = ac$,

$$c^6 + 3\left(\frac{1}{a} + 1 + a\right)c^4 - \left(\frac{1}{a^3} + \frac{1}{a^2} + \frac{1}{a} - 1 + a + a^2 + a^3\right)c^2 - \left(\frac{1}{a^2} + \frac{1}{a} + 1 + a + a^2\right) = 0 \qquad (9).$$

Put $\frac{1}{a} + a = t$,

$$c^6 + 3(t+1)c^4 - (t^3 + t^2 - 2t - 3)c^2 - t^2 - t + 1 = 0 \qquad (10).$$

Put $c^2 = x - 1$,

$$x^3 + 3tx^2 - t(t^2 + t + 4)x + t^3 = 0 \qquad (11).$$

Put $x = yt$,

$$ty^3 + 3ty^2 - (t^2 + t + 4)y + t = 0 \qquad (12).$$

Put

$$y = \frac{1 - z}{1 + z} \qquad (13),$$

$$t = 2\frac{(\sqrt{Z} + 1)^2}{(1 - z)(1 + z)^2}, \quad Z = -z + z^2 + z^3 \qquad (14),$$

$$x = 2\frac{(\sqrt{Z} + 1)^2}{(1 + z)^3} \qquad (15),$$

$$c^2 = \frac{1 - 5z - z^2 + z^3 + 4\sqrt{Z}}{(1 + z)^3} \qquad (16),$$

$$t - 2 = \frac{4\sqrt{Z}(\sqrt{Z} + 1)}{(1 - z)(1 + z)^2} \qquad (17),$$

$$t + 2 = \frac{4(\sqrt{Z} + 1)}{(1 - z)(1 + z)^2} \qquad (18),$$

$$t^2 - 4 = \frac{16\sqrt{Z}(\sqrt{Z} + 1)^2}{(1 - z)^2(1 + z)^4} = \left(\frac{1}{a} - a\right)^2 \qquad (19).$$

$$C^2 = 64z^2\sqrt{Z}\,\frac{(1 + z^2)(1 + 2z - 6z^2 - 2z^3 - z^4) - 8z^2\sqrt{Z}}{(1 - z^2)^5(1 + 4z - z^2)}$$

$$= \frac{64z^2\sqrt{Z}}{(1 + z^2)(1 + 2z - 6z^2 - 2z^3 + z^4) + 8z^2\sqrt{Z}} \qquad (20),$$

as in (2), § 42, for $\mu = 20$.

To connect up the results, $\mu = 20$ and 40,

$$\frac{s(\omega) - s(10v)}{s(\omega) - s(2v)} = a^2 \text{ in } \mu = 40 \qquad (21),$$

$$\frac{s(\omega) - s(5v)}{s(\omega) - s(v)} = \left(\frac{\epsilon + p}{\epsilon - p}\right)^2 \text{ in } \mu = 20 \qquad (22),$$

are equal, so that

$$a_{40} = \frac{p + \epsilon}{p - \epsilon} = \frac{a_{20} + b_{20}}{a_{20} - b_{20}} \qquad (23),$$

and thus, with $a_{20} = a$, $b_{20} = b$,

$$\frac{a^2 + b^2}{a^2 - b^2} = \tfrac{1}{2} t = \frac{1 - z + z^2 + z^3 + 2z\sqrt{Z}}{(1 - z)(1 + z)^2} \qquad (24),$$

$$\frac{b^2}{a^2} = \sqrt{Z} \qquad (25),$$

$$a_{20} = \frac{1}{z_{40}} \qquad (26)$$

(KIEPERT, 'Math. Ann.,' 32, p. 119), and the preceding results are thus merely bisection formulas for $\mu = 20$.

We arrive at the conclusion that it is the quotient of two theta functions, θu and $\theta(u - v)$, with constant phase difference v, which is required in dynamical application, the functions α, β, γ, δ for instance employed by KLEIN in top-motion; but the separate theta function θu has no mechanical interest.

This quotient, qualified by the constant factors $\theta 0$ and θv, is an elliptic function of u when v is a half-period, dn u for instance when v is the half-period K, and the quotient is the μ^{th} root of an algebraical function of the elliptic functions of u when v is an aliquot μ^{th} part of a period; in this way we express the result of ABEL's pseudo-elliptic integral.

The formation of this algebraical function for the simplest values of μ has been our chief object, and in the course of the work the ellipsotomic problem has been carried out of the determination of the Division-Values of the Elliptic Function.

The Transformation problem may be considered solved at the same time by means of symmetric functions of the division-values; but as Transformation has no dynamical utility, it has not been developed in this memoir.

Electric Arc.

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X. *On the Resistance and Electromotive Forces of the Electric Arc.*

By W. DUDDELL, *Wh.Sc.*

Communicated by Professor W. E. AYRTON, *F.R.S.*

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[PLATE 2.]

SINCE DAVY'S discovery of the electric arc, a century ago, down to the present time, the nature of the physical processes going on in it, and the mechanism by which it conducts electricity, have been the subject of almost uninterrupted discussion and experiment. In order to explain the fact that the equation connecting P.D. current and length appears to contain a large practically constant term, experimenters have assumed that the arc possesses resistance and E.M.F., though which of the two is the more important in obstructing the flow of the current, or whether both must be considered, has been, and is still, a matter of controversy, the settlement of which, it is hoped, will be furthered by the experimental results described in this communication.

A priori it is highly probable that the resistance and E.M.F.'s of the electric arc, if they exist, will be functions of the current; it is therefore necessary to first consider the definitions of these quantities, as it will largely depend on the definitions adopted whether the arc can be said to possess a resistance, an E.M.F., or neither. The ordinary text-book definitions of resistance and E.M.F. generally start with the assumption that they are constant quantities independent of the current flowing, and their possible variation is generally developed as a secondary effect due to the current altering the state or nature of the body or apparatus considered, these alterations in the state being the primary cause in the change in resistance and E.M.F. observed.

There is much experimental evidence to support the view that when a steady current, A, flows through *any* conducting apparatus, the potential difference, V, between its terminals can be written $V = E + RA$ when E and R, the E.M.F. and the resistance, only depend on the nature, state and movement of the apparatus, and are not directly functions of the current A or the potential V; so that the equation connecting the P.D. and current for any apparatus *under perfectly constant conditions* only contains the current raised to the powers 0 and 1, and the coefficients of these terms are considered as constant specific properties of the conducting apparatus under the given conditions. It is, however, quite conceivable that conductors might exist for which the equation might contain other powers of the current, and in this case their coefficients would be equally justly considered as specific properties of the conductor for which at present no names exist. Is the arc such a conductor ?

On the assumption that *when the conditions are maintained constant* the P.D. can be represented by an equation of the form $V = E + RA$, for all values of the current, then the power spent or furnished by the apparatus $VA = EA + RA^2$, that is, it consists of two parts—the one depending on A^2, and therefore irreversible, so that the sign of the power does not change with change of sign of the current, and the other depending on the first power of the current, and therefore a reversible phenomenon, so that if the apparatus absorbs energy when the current flows in one direction, it will give back energy when the current flows in the opposite direction.

This idea of distinguishing E.M.F.'s from resistances, according to whether the dissipation or absorption of energy is a reversible or irreversible phenomenon, is by no means new, as it underlies the views expressed by Professor FITZGERALD[*] and GRAY[†] in the letters they contributed to the 'Electrician' in the discussion of Messrs. FRITH and RODGERS' paper,[‡] and has also been suggested by Professor S. P. THOMPSON. It seems to afford a satisfactory basis for a definition of resistance and E.M.F., which will be adopted in this communication.

Definition.

Suppose any apparatus under any given set of conditions, through which a certain current is flowing, and that it is required to determine its resistance and E.M.F. under these particular conditions and for this particular current. The energy transferred electrically between the source and the apparatus can be divided into two parts : the one an irreversible part, so that if the direction of the current be conceived reversed the direction of the transfer of energy remains unchanged, and the other a reversible part. If it be found that the irreversible transfer of energy is proportional to the square of the current, and the reversible to the first power of the current, when in some way or other perfect constancy is maintained in all the conditions of the apparatus, such as the size, shape, nature, temperature, temperature gradients, relative movements, &c., of the different parts of the apparatus which are existing with the particular current and under the given set of conditions, then the irreversible rate of transfer of energy divided by the square of the current will be defined as the resistance, and the reversible rate of transfer of energy divided by the first power of the current will be defined as the E.M.F. of the apparatus, under the given set of conditions and for that particular current which was initially supposed to be flowing.

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[*] 'The Electrician,' 1896, vol. 37, pp. 386, 489.

[†] 'The Electrician,' 1896, vol. 37, p. 452.

[‡] 'Proc. Phys. Soc.,' 1896, vol. 14, p. 307.

In this definition the qualification, "if the irreversible transfer of energy is proportional to the square of the current and the reversible to the first power of the current," predetermines that the apparatus must obey OHM's law over whatever range it may be possible to vary the current *without in any way changing the state* of the apparatus, in order that it can be said to have a resistance at all. If, however, the qualification is in any case not fulfilled, it will become necessary to consider the terms in the conceivable equation between V and A other than those in which A occurs to the powers 0 or 1.

So far nothing has been said of the signs which the two quantities resistance and E.M.F. can have, as their signs are more or less a matter of convention. If we call a current flowing round the circuit in the same direction as the E.M.F. of the source would tend to make it flow a + current, and a transfer of energy from the source to the apparatus a + transfer of energy, then their signs are determined and agree with ordinary practice, so that the resistance and the E.M.F. of the apparatus which oppose the flow of the current will have + signs. It is to be noticed, however, that this definition does not preclude in any way the possible existence of a negative resistance; for if, instead of an irreversible transfer of energy from the source to the apparatus, proportional to the square of the current, there were found (the conditions of the apparatus being, of course, maintained constant as before) to be a transfer in the opposite direction, *i.e.*, from the apparatus to the source, then the coefficient of A^2 would have to be negative, so that in this case the apparatus would possess a true negative resistance. Although in what follows it will be shown that this is not the case with the arc, it is as well to draw attention to the matter, as a considerable part of the controversy on the negative resistance of the arc under certain conditions arose from some of those who took part defining resistance as an essentially positive quantity, and then trying to prove that it could not be negative in the case of the arc.

A single value of V corresponding to a single value of A is evidently not sufficient to determine whether any conductor fulfils the above definition of resistance and E.M.F. To determine this the current must be varied over some range, δA, and in such a manner that the conditions of the conductor remain unchanged, and a series of observations must be made within this range.

The essential stipulation, that the test must not alter the body tested, is the main difficulty in the experiments on the resistance and E.M.F.'s of the arc. For it is well known that, corresponding with each steady value of the current, the size and configuration of the vapour column and craters are different in spite of the fact that the length, the nature of the electrodes, and the other conditions may be kept constant, so that the arcs corresponding with any two different steady values of the current, however nearly equal they may be, are really two distinct and different phenomena. Therefore all methods which depend on the steady change δV in the potential difference V produced by a given steady change δA in the current A, that is to say, which depend on an excursion on the steady curve between V and A, however small it may be, simply measure the difference between the P.D.'s required to maintain an arc with a current A and a distinct and different arc with a current $A \pm \delta A$, which is evidently no measure of either the resistance of the arc with current A or with $A \pm \delta A$.

If the measuring current δA is only applied for a short time δt, it is necessary that the energy supplied to or removed by it shall be so small as not to appreciably alter the thermal conditions of the very small mass of gaseous and other material which is taking part in the conduction of the current. How extremely short the time that may elapse is will appear later; for the present it is sufficient to point out that it has been found that even in $\frac{1}{10,000}$ second a change of 3 per cent. in the arc current has appreciably altered the thermal conditions and the light emitted by the arc.*

It thus appears that the only available methods of experimentally determining the resistance and E.M.F. of the arc must be based on making the necessary change in the main current, *i.e.*, the measuring current, as small as possible, and on completing the test so soon after making this change that none of the conditions of the arc will have had time to appreciably alter before it is completed.

The first method tried consisted in sending the oscillatory discharge from a condenser through the arc, and recording by means of an oscillograph the variations in the P.D. between the terminals of the arc and in the current through it. If the frequency of the oscillatory discharge can be made so high that the conditions of the arc are not in any way altered by it, then the wave-forms of the oscillatory part of the P.D. and current will be similar curves and in phase if the arc possesses a true resistance. This was not found to be the case with the oscillations used, which had frequencies up to 5000 ∽ per second, the current oscillation always lagging behind the P.D. oscillation.

At low frequencies and with solid carbon electrodes the oscillations were 180° out of phase, and this difference was gradually reduced with increase in frequency to below 90° at 5000 ∽ per second, and there were indications that this lag would finally disappear if a much higher frequency were used, so that the conditions of the arc were not altered by the oscillations.

A large number of experiments, some of which have been published in a paper† before the Institution of Electrical Engineers, were made to determine the effect of small rapid variations in current on the conditions of the arc itself. The conclusion drawn from the above experiments was that a very much higher frequency than 5000 ∽ per second was necessary in order that the arc might not be affected by the measuring current.

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* 'Journal of the Institution of Electrical Engineers,' 1901, vol. 30, p. 236.

† *Ibid.*

Owing to various reasons, the above method was not suitable for these higher frequencies; consequently a new method was devised similar to that used by Messrs. Frith and Rodgers,* based on the R.M.S. values of the superimposed P.D. and current, and not on the instantaneous values of these quantities, with this difference from Messrs. Frith and Rodgers' method, that there was a criterion, when a result was obtained, as to whether the arc was behaving like an ordinary resistance or not.

Basis of Method Adopted.

Consider any apparatus A, fig. 1, which may have resistance and E.M.F. but no self-induction or capacity, through which a steady direct current may be flowing, and let there be mixed with the direct current an alternating testing current of R.M.S. value C.

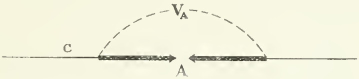

Fig. 1.

Let V_A be the R.M.S. value of the alternating part of the P.D. between the terminals of A, and let v_A and c be the instantaneous values of these latter quantities.

The impedance of the apparatus A $= \sqrt{\left(\frac{1}{T}\int_0^T v_A^2\,dt\right)} \Big/ \sqrt{\left(\frac{1}{T}\int_0^T c^2\,dt\right)} = V_A/C = I_A$.

Suppose that the frequency of the alternating current can be made such that the conditions of the apparatus are not in any way changed by the alternating current, then if the apparatus has a true resistance it will be a constant, so that the instantaneous values v_A and c will have a constant ratio, *i.e.*, will obey Ohm's law. Then the wave-forms of V_A and C will be similar curves and in phase, and the true resistance of the apparatus $= v_A/c = V_A/C = I_A$.

A criterion is now required that v_A and c do obey Ohm's law, and this is supplied by the power-factor of the apparatus A, the power-factor being defined as

$$\frac{1}{T}\int_0^T v_A c\,dt \Big/ \sqrt{\left(\frac{1}{T}\int_0^T v_A^2\,dt\right)} \sqrt{\left(\frac{1}{T}\int_0^T c^2\,dt\right)}.$$

For it can be proved that the necessary and sufficient condition that the power-factor may be unity is that the wave-forms of V_A and C are similar curves and in phase, so that v_A and c obey Ohm's law.

Therefore if, when the current C and its frequency are such that the conditions of the apparatus are not changed, it can be proved that the power-factor of A is unity, then A has a true resistance numerically equal to I_A. Further, in any apparatus in which the resistance is a function of the conditions, the possibility of obtaining the power-factor unity is a proof of the constancy of the resistance and consequently of the conditions, so that if the apparatus A is an arc, and if it can be shown that a sufficiently high value of the frequency can be reached for which the power-factor is unity, then the conditions of the arc are not being altered by the alternating testing current, and the arc has a true resistance numerically equal to I_A.

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* 'Proc. Phys. Soc.,' 1897, vol. 14, p. 307.

It is assumed above that the arc or apparatus A has no self-induction or capacity; to prove this it must be shown that not only can the frequency be increased till the power-factor becomes unity, but also that it remains so for a considerable further increase of frequency.*

Finally, therefore, in order to prove that the arc has a true resistance, and to find its value, it is necessary to show:—First, that it is possible to find a value of the frequency of the alternating testing current for which the power-factor of the arc with respect to this current is unity; second, that the power-factor remains unity and the impedance constant even when the frequency is greatly increased above this value; third, to determine the value of the impedance of the arc under these conditions, which will also be its true resistance.

Method of Measuring the Impedance and Power-Factor.

At first sight it would seem as if there were a considerable number of available methods for accurately measuring these quantities. But the number of methods becomes exceedingly limited when it is considered that it is necessary for the alternating testing current C to have as small a R.M.S. value as possible (0·1 ampère was that generally used in the experiments), and that the effects due to this small current have to be sorted out when it is mixed with a direct current of 10 ampères or more. Added to this, to make the difficulties greater, it was finally found necessary to use frequencies up to and even over 100,000 ~ per second. Wattmeters and dynamometers were tried and abandoned, and finally the well-known 3-voltmeter method† was adopted.

A non-inductive resistance R was placed in series with the apparatus A (fig. 2), through both of which the main direct current flowed; to this direct current there was added, as before, an alternating measuring current of R.M.S. value C.

Let V_A, V_R, and V be the R.M.S. values of the alternating part of the P.D.'s as shown in fig. 2. The impedance of A is $I_A = V_A/C = RV_A/V_R$. Power factor of A is $P_A = (V^2 - V_A{}^2 - V_R{}^2)/(2V_AV_R)$.

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* It seems possible that the power-factor of a conductor which did not possess self-induction or resistance in the ordinary sense of these terms might still depart from unity at very high frequencies, owing to the time taken by the carriers of the electric charge to hand it on becoming comparable with the periodic time of the testing current.

† See AYRTON and SUMPNER, 'Roy. Soc. Proc.,' vol. 49, p. 424.

The two quantities, the impedance and power-factor, are therefore determined in terms of a resistance R and three R.M.S. voltages quite independent of any

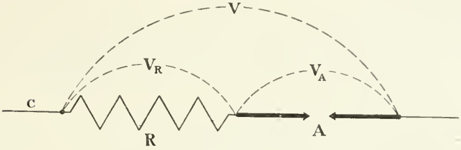

Fig. 2.

assumptions as to the wave-form of the alternating testing current. If the same voltmeter is used to measure each of these voltages, then it will be noticed that the results only depend on the relative calibration of one instrument, a consideration of great importance owing to the difficulties in the way of accurate absolute calibration with the very high frequencies used.

Circuit and Apparatus Used.

In order to measure the impedance and power-factor by the method just considered, several different arrangements of the circuit were tried; that finally adopted for the

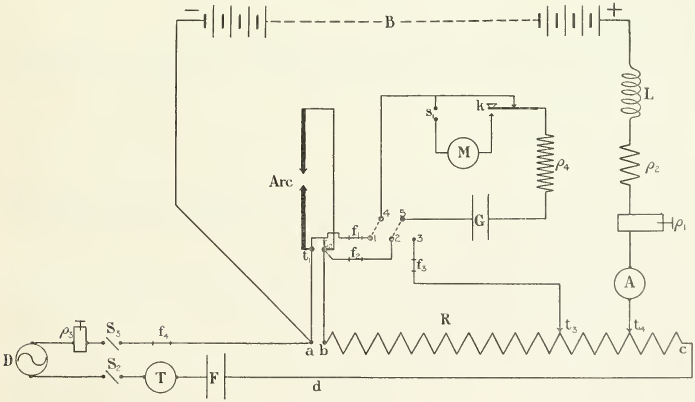

Fig 3

experiments is shown diagrammatically in fig. 3. The main direct-current circuit consisted of:—

B, a battery of from 50 to 90 accumulators which supplied the direct current to the arc,

L, a self-induction,

ρ_1, ρ_2, adjustable resistances,

A, a Weston ammeter which indicated the direct current through the arc,

R, the standard non-inductive resistance, consisting of 12 coils of about 0·5 ohm each, described later,

t_1 and t_2, the terminals of the arc lamp,

t_3 and t_4, terminals which could be moved along R,

1, 2, and 3, mercury cups in a wax block,

f_1, f_2, and f_3, fine fuses in the connections between the above points.

The direct P.D. between the terminals of the arc lamp (called "P.D. arc lamp") was measured with a Weston voltmeter (not shown) which was connected between the points 1 and 2. The same voltmeter, which could be connected between points 2 and 3, gave direct P.D. between the terminals t_2 and t_3 (called "P.D., R"), from which the resistance of R was obtained in terms of the readings of the voltmeter and ammeter, both of which were carefully standardised.

An image of the arc was projected by means of a lens on to a screen, divided and marked so as to read the actual distance in millimetres between the *ends* of the carbons; this distance is called the "arc length."

The circuit for adding the alternating testing current to the direct current consisted of:—

D, a special high-frequency alternator which supplied the testing current, capable of producing small alternating currents having frequencies up to 120,000 ∼ per second,

ρ_3, a variable resistance,

S_2 and S_3, plug switches,

f_4, a fine fuse,

T, an Ayrton and Perry reflecting twisted-strip ammeter, having a sensibility of 400 scale divisions for 0·1 ampère at a scale distance of 1700 divisions (1 division = $\frac{1}{40}$ inch),

F, a condenser to prevent any direct current from flowing through the alternator, the capacity of which was 1 mf. for the frequencies from 120,000 to 50,000 ∼ per second; 2·5 mf. for frequencies from 50,000 to 2000 ∼ per second; and increased up to 15 mf. at 250 ∼ per second.

The alternating current supplied to the arc circuit was kept at a constant R.M.S. value as read on T by means of the adjustable resistance ρ_3; this current flowed through the arc and R in series, and was practically prevented from flowing through the battery by the self-induction L. At the higher frequencies of from 10,000 to 120,000 ∼ per second this self-induction behaved almost like an insulator; at the lower frequencies a small percentage of the alternating current flowed round the battery side of the circuit; but this did not affect the accuracy of the results, as the expressions for the impedance and power-factor are independent of the current, so that it is only necessary to maintain the testing current constant during each test.

The circuit for measuring the alternating part of the P.D. arc lamp V_A, the P.D. between the terminals of R, V_R, and the P.D. total, V, consisted of:—

M, a thermo-galvanometer, whose deflections were practically proportional to the mean squared value of the current through it, and which, though practically non-inductive, gave a deflection of about 500 scale divisions for 1 milliampère of *alternating* current. The deflections of this instrument were read on the same scale as those of T, so that both could be observed at one time;

ρ_4, an ordinary resistance box,

k, a key,

S_1, a switch, consisting of mercury cups in a wax block,

G, a condenser which allowed a current due to the alternating part of the P.D. to flow through M, but prevented any current due to the direct P.D. from flowing through it,

4 and 5, mercury cups by means of which the measuring circuit could be connected to either the points 1 and 2; 2 and 3; or 1 and 3; to measure V_A, V_R, and V respectively.

The impedance of the measuring circuit, which consists of the thermo-galvanometer M, the resistance ρ_4, and the condenser G, need not be accurately known, as it is only the relative values of V_A, V_R, and V which are required to a high degree of accuracy and not their absolute values. As the frequency in each experiment was kept constant to well within 1 per cent., the impedance of the measuring circuit need not be quite independent of the frequency.

In all arcs neither the direct P.D. nor the current keep quite steady, owing to the necessity of feeding the carbons together, and to the impurities, cracks, &c., existing in them, so that the comparatively slow variations produced must be prevented from sending any appreciable currents through M; for this reason the capacity of G was made as small as compatible with the impedance of the measuring circuit, not depending too much on the frequency. A standard $\frac{1}{3}$ mf. condenser was used for G for all frequencies from 120,000 to 10,000 $\sim$ per second inclusive; down to 3,000 1 mf. was employed, and for all lower frequencies 5 mf. Even under these conditions it was absolutely essential that the battery B should not be in use for any other experiments, or spurious currents were obtained through M, and such a thing as the arc giving a small hiss sent the spot off the scale.

The key *k* was so arranged that in its up position the condenser G was always kept charged to the correct P.D., so that on depressing it G was neither suddenly charged nor discharged through M, for owing to the delicate nature of the latter any considerable sudden change in the P.D. between the armatures of G sent sufficient current through M to burn it up. Even this precaution did not prevent M from being burnt up several times, owing to the battery connections being broken or the arc going out while the key k was depressed.

The switch S_1 was to enable M to be completely disconnected when taking its zero, as it was thought that at these high frequencies, so long as one pole was connected, there might be a small current flowing into the instrument, due to the capacity of the instrument with surrounding objects, but this effect was not observed.

Owing to the very high frequencies used, very great care had to be taken in arranging the circuit so as to avoid self-induction and capacity errors in the leads and connections. All the leads through which the alternating current flows were carefully twisted and bound together.

To reduce any possible error caused by the lead t_2b between the arc and R, the drop along which had to be included either with the arc or with R, this lead was made by twisting together 12 No. 23 double cotton-covered wires, to avoid possible skin effects, and its length was reduced to 60 centims. As first constructed, each wire of the lead t_2b was twisted with the corresponding wire of lead t_1a, which was of the same length and made in the same way, and then the 12 pairs of wires were twisted together. It was found on testing these leads that at a frequency of 18,000 ~ per second, and with a P.D. of 33 volts between the two leads, an alternating current of about $1{\cdot}9 \times 10^{-3}$ ampère flowed between them due to their capacity. To reduce this capacity current, two new leads were constructed, exactly the same as before, only that instead of twisting the individual wires belonging to each lead together in pairs, all the wires belonging to each lead were stranded together so as to form two separate leads. The lead t_2b was then bound over with a layer of silk tape and the two leads were twisted together. On re-testing in the same way as before, the capacity current was found to be reduced to about $0{\cdot}53 \times 10^{-3}$ ampère with 33 volts between the leads. As the alternating P.D. between the leads was under 0·5 volt in most of the experiments, this capacity current was negligible compared with the working current of 0·1 ampère.

As the self-induction of R had to be determined and allowed for, the lead t_2b was included with R, so that its small self-induction could be corrected for at the same time.

The lead c, d was brought back along the connections b, c between the 12 coils of R, so as to neutralise as well as possible the magnetic field of these connections.

In arranging the circuit the capacity of those parts of the main circuit between the measuring points t_1, t_2, t_3, as well as of the whole of the measuring and alternator circuits, to surrounding objects and to earth, was kept as small as possible, so as to avoid what might be called capacity leaks. The alternator itself was practically insulated from earth by being fixed down to a wooden frame, and the field circuit of the alternator was well insulated and removed from earth.

To form some idea of the magnitude of these capacity currents or leaks, the four points *a*, *b*, *c* and the arc, were opened in pairs. One of the opened points was re-connected through the thermo-galvanometer used as an ammeter to measure the capacity current supplied to that part of the circuit between the instrument and the other point that was opened. The direct current circuit was disconnected at the points *a* and *d*. The alternating currents observed are given in Table I. for a R.M.S. P.D. of 3·65 volts and a frequency of 100,000 ∼ per second.

Table I.—Capacity Currents in Leads.

Thermo-galvanometer at—	Points.				Alternating current in 10^{-4} ampère.
	a.	*b*.	*c*.	Arc.	
a	—	closed	closed	open	3·8
a	—	,,	open	closed	1·6
b	closed	—	closed	open	2·9
b	,,	—	open	closed	1·2
c	,,	closed	—	open	3·9
c	,,	open	—	closed	1·2

The maximum value of the capacity leak observed is $3·9 \times 10^{-4}$ ampère at the highest frequency used in any series of experiments, and at a P.D. about seven times as high as that used, so that if the capacity current is proportional to the P.D., it should not exceed $0·6 \times 10^{-4}$ ampère, or about 0·06 per cent. of the working current of 0·1 ampère, and may therefore be neglected. Even if this capacity leak had been many times larger, it would not have appreciably affected the P.D.'s measured, since at these high frequencies the arc behaves like a non-inductive resistance, and therefore the measuring current and the capacity current would add approximately as vector quantities at right angles.

As it is only the relative values of V_A, V_R, and V that are required very accurately, any small self-induction in the leads connected with the points 1, 2, and 3, or in the measuring circuit itself, is of no importance. Nevertheless, all the leads were carefully stranded together to prevent any E.M.F.'s being induced in this circuit, caused by magnetic induction. The only wire which could not be stranded with a corresponding wire was about 30 centims. of the lead between the fuse f_3 and the movable contact t_3. Experiments were made by varying its length and position to see if it introduced any error, but none could be detected.

The condensers F and G were placed some distance apart, so as to prevent any direct electrostatic action between them. This, as well as any mutual induction, both electrostatic or magnetic, between any part of the main or alternator circuits and the measuring circuit was examined for, but none could be detected. Experiments were also made to see if the condenser G really prevented M from being deflected by steady direct P.D.'s; for this purpose steady P.D.'s of from 100 to 180 volts were applied to the terminals of the arc, the carbons being separated, and M was connected in the ordinary way as if to measure V_A. A deflection of M might have occurred due to leakage or electrostatic forces, but no such deflection was observed.

The Arc Lamp.

A hand-fed arc lamp, enclosed in an iron case, was used in all the experiments. The sliding contacts were shunted by flexible leads, so as to avoid any uncertainty in the resistance of these contacts. The resistance of the contacts and of the carbons was determined by adjusting the carbon holders to 10 centims. apart as in use, short-circuiting them with a brass rod and the different carbons in turn, and then measuring the drop in volts between the terminals of the lamp when 10 ampères direct current flowed round the frame and holders, etc. The results are given below :—

TABLE II.

Carbon holders short-circuited with	Resistance.
	ohms.
Brass rod	0·0017
11 millims. solid "Conradty Noris" carbon	0·13
11 ,, cored ,, ,, ,,	0·22
11 ,, solid "Le Carbon" electrographitic carbon	0·048

The self-induction of the loop formed by the frame of the lamp with the carbon holders 10 centims. apart and short-circuited by the brass rod was approximately determined by passing an alternating current round the lamp and measuring the P.D. between its terminals, at a frequency of 30,000 ~ per second. The value obtained was $2·4 \times 10^{-7}$ henry, which was used as a correction. All the above tests were taken with the lamp in place in its case, and with the carbon holders, etc., in exactly the positions they occupied when commencing a test on the impedance of the arc.

The Standard Non-Inductive Resistance, R.

The essence of the test of the arc consisted in comparing its behaviour to alternating currents of various frequencies with that of the standard resistance R in series with it. It was, therefore, necessary that R should be as free as possible from self-induction and capacity. The type of resistance adopted was that described by Professor AYRTON and Mr. MATHER before the Physical Society* in 1891.

As both the self-induction and the capacity depend on the size of the resistance, it was decided to make the resistance as small as possible and to allow the temperature of the strips to rise considerably, which necessitated slightly modifying their design. The resistance consists of 12 platinoid strips, each about 170 centims. long, 2·5 centims. wide, and 0·076 millim. thick. Each strip is folded back on itself and has its ends soldered to two brass blocks let into the top of the frame, and is stretched tight with a tension of about 4 lbs. by means of a brass spring attached to a small glass tube, about 5 millims. diameter, at the bottom of the loop formed by the strip. Between the up and down sides of the strips is placed a sheet of asbestos millboard, about $\frac{1}{32}$ inch thick, and the strips are pressed together against this by glass rods from side to side of the frame.

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* 'Philosophical Magazine,' 1892, vol. 33, p. 187.

The resistance of each of the 12 strips was roughly adjusted to 0·5 ohm, and the strips could be used in series or parallel by connecting up the brass blocks forming the ends of the strips with copper links and set-screws as required. Owing to the considerable heating of the strips by the current, their resistance depended on the current; thus the resistance of all the strips in series, which was 6·00 ohms with 1 ampère flowing, rose to 6·25 ohms with the current of 10 ampères which was used in many of the experiments. For this reason, and because it formed a check on the instruments, the resistance of R was determined during each experiment from the known values of the direct current and P.D. between its terminals, and the value so obtained was used in calculating the results.

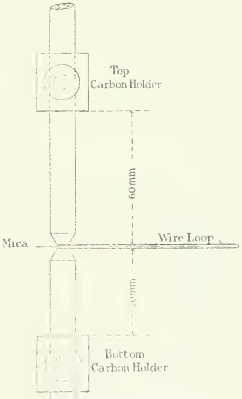

Fig. 4.

The apparent self-induction of R, including the connection between it and the arc lamp already mentioned, was measured by comparing it with a non-inductive resistance put in place of the arc. This latter resistance (see fig. 4) was made to imitate an arc possessing non-inductive resistance localised between the ends of the carbons. It consisted of 158 millims. of No. 38 platinoid wire bent back on itself, the two extremities being soldered to the ends, previously copper plated, of two solid "Conradty Noris" carbons; these carbons were held in the carbon holders of the lamp so that the resistance wire occupied the position the arc would when burning. A piece of mica was interposed between the ends of the carbons which served to keep the loop of wire taut. This method of determining the correction to be applied to R really converts the test of the arc into a substitution test, for having determined how an ordinary metal resistance behaves when localised between the carbon tips, the behaviour of an arc substituted for it under exactly similar conditions was compared with it.

To compare R with the wire, they were adjusted to have practically the same resistance. A current, either direct or alternating, having a R.M.S. value of 0·1 ampère as indicated by T, was sent through R and the wire in the place of the arc in series, fig. 3. The P.D. between the terminals of R and between the terminals of the arc lamp (*i.e.*, wire) was measured by means of M, in the same way as in the experiments on the arc, the condenser G being short-circuited. The amount by which the impedance of R exceeded the impedance of the arc lamp and wire is tabulated below for the strips 1 to 7 having a resistance of 3·50 ohms. Each of the results is the mean of at least 12 comparisons. The self-induction of the loop formed by the frame of the arc lamp has been found to be about $2\cdot4 \times 10^{-7}$ henry, and the self-induction of the wire itself was calculated to be about 3×10^{-8} henry; so that the total self-induction of the lamp and wire is $2\cdot7 \times 10^{-7}$ henry. Assuming that the alternator gives a sine-wave form, which was approximately the case, and allowing for this self-induction and for the difference in resistance 0·03 per cent. between the lamp and R, the true power-factor, $\cos \eta$ of R, has been calculated, from which η the lag of the current in R behind the P.D. and the time constant have been deduced.

TABLE III.—Test of R, Resistance 3·50 ohms.

Frequency ∼ per second.	Impedance of R > impedance of lamp and wire.	$\cos \eta$.	η.		Time constant in 10^{-7} second.	Impedance of R > its resistance.
	per cent.		°	′		per cent.
0 (direct current)	0·03	1·0000	0		—	0
32,400	0·17	$0\cdot998_5$	3	${}_8$	$2\cdot{}_7$	$0\cdot1_5$
41,000	0·25	$0\cdot997_6$	5	5_8	$2\cdot{}_7$	$0\cdot2_4$
50,700	0·39	$0\cdot996_1$	5	${}_4$	$2\cdot{}_8$	$0\cdot3_9$
60,000	0·51	$0\cdot994_8$	5	5_1	$2\cdot{}_7$	$0\cdot5_2$
80,000	0·90	$0\cdot990_6$	7	5_2	$2\cdot{}_8$	$0\cdot9_4$
100,000	1·28 {mean of 24 tests}	$0\cdot986_3$	9	3_0	$2\cdot{}_7$	$1\cdot3_7$
120,000	1·85	$0\cdot980_1$	11	2_7	$2\cdot{}_7$	$1\cdot9_9$

Method of Experiment.

The arc length, the direct current, and the frequency of the alternating testing current having been decided upon for any experiment, the test was carried out in the following manner. The carbons having burnt into shape corresponding with the required current and length, a rough experiment on the impedance of the arc was made, and the value of R adjusted by moving t_3 (fig. 3) to that contact on R which made V_A and V_R most nearly equal, as this gives the greatest accuracy in the power-factor by the 3-voltmeter method.

The main direct current was now interrupted and the arc short-circuited by pressing the carbons together. The testing current was adjusted to its working value, in most cases 0·1 ampère, as read on T, and the resistance ρ_1 in series with M was adjusted until V_R gave a deflection of 100 scale divisions on M. The reason that this adjustment was made without the direct current flowing was that it formed a check on the satisfactory working of the measuring circuit, since any apparent change in the sensibility of M when the direct current was re-established would have indicated an error somewhere.

With solid carbons the positive or upper electrode was adjusted to project 6 centims. from its holder and the lower or negative 4 centims. With cored carbons these lengths were 7 and 4·5 centims. respectively. The object of adjusting these lengths was to make the mean resistance and self-induction of the loop formed by the frame of the arc lamp and carbons as nearly as possible the same in every experiment.

The arc was now re-started, and the length and direct current having been adjusted, the carbons were fed together as they burnt away, so as to keep the direct current constant during the whole of the time (about half an hour) that V_A, V_R, and V were being determined. This kept the P.D. arc constant, as long as the P.D. of the battery remained constant; any slight drop in this P.D. was compensated for by adjusting ρ_1, any considerable drop necessitated recommencing the experiment. The arc length with solid carbons also remained constant; but with cored carbons it constantly varied about a mean value according to the amount of material from the core present in the arc. In all cases, readings were only taken when the length was observed to be correct as well as the direct P.D. and current.

As soon as about 5 millims. had burnt off the end of the positive carbon, the deflections of M corresponding to V_A, V_R, and V were observed in turn, the zero of M being taken after each reading, until in most cases five consecutive sets were obtained which were reasonably consistent with one another, the R.M.S. value of the testing current as read by T and its frequency being kept constant. It was easy to obtain individual deflections corresponding with V_R which differed from the mean by less than 0·3 per cent. The deflections corresponding with V_A and V were not so definite, V_A being within 1 per cent. and V within 0·6 per cent. of the mean, except in a few exceptionally unsteady arcs, such as long-cored arcs and small-current arcs.

The values of the direct P.D. arc lamp and direct P.D. R were noted, and the drop of volts in the frame of the lamp and carbons was found by pressing the carbons together, the direct current being so adjusted that when the carbons were in good contact its value was that used for the experiment. This observation was repeated until consistent results were obtained with the carbons hot as in use. By deducting this value from P.D. arc lamp, P.D. arc was obtained.

The relative calibration of the thermo-galvanometer M was then determined by means of direct currents. This completed the observations required for a single experiment.

Mean deflections corresponding with V_A, V_R, and V, were calculated and corrected for the relative calibration of the thermo-galvanometer. From these values the impedance of the arc lamp $I_A = I_R, V_A / V_R$, and the power-factor of the arc lamp, $P_A = (V^2 - V_A^2 - V_R^2)/2V_AV_R$, were calculated.

In order to obtain from these values the impedance and power-factor of the *arc itself*, a small correction had to be applied to I_A for the resistance and self-induction of the loop formed by the frame of the lamp and the carbons, and also to P_A for the self-induction of R which had previously been determined. To make these small corrections, it was necessary to assume that the alternating current had a sine-wave form, which was approximately the case.

As a check on the method of experiment and on the calculation and correction of results, the impedance and power-factor of the platinoid resistance, described on p. 317, which had a resistance of 3·499 ohms and a self-induction of about 3×10^{-8} henry, were determined, the experiment and calculations being performed in the same manner as for the arc. The values obtained were: impedance 3·50 ohms, power-factor 0·999, which show that the method was satisfactory in this case.

Results Obtained by Varying the Frequency.

The fundamental experiment of this investigation into the resistance of the electric arc consists, as has already been explained, in varying the frequency of the superimposed alternating testing current, in order to determine whether with a sufficiently high frequency the condition of the arc will remain unchanged, the value of the resistance being then measured at this frequency. The criterion that the conditions of the arc remain unchanged has been shown to be that the power-factor of the arc as measured with the superimposed alternating current must be unity. The true resistance will then be equal to the impedance.

The results of the experiments on the effect of varying the frequency on the power-factor and the impedance for solid and cored* arcs are represented graphically in Curves I. and II. (Plate 2).

With solid carbons the power-factor at 250 ~ per second is $-$ 0·91. On increasing the frequency it decreases numerically until it vanishes and changes sign at 1,950 ~ per second, the waves of superposed alternating P.D. and current being then 90° out of phase. With further increase of frequency the power-factor increases rapidly at first, then more and more slowly, becoming asymptotic to $+$ 1, and finally practically attains this value at a frequency of 90,000 ~ per second; above this frequency the power-factor is, within the limits of experimental error, equal to $+$ 1 up to the highest frequency attained, namely, 120,000 ~ per second. The impedance of the solid arc increases with increase of frequency from 0·97 ohm at 250 ~ to 3·8 ohm at a frequency of 90,000 ~ per second, above which it remains practically constant.

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* "Solid" and "cored" arc mean respectively arc between two solid carbons and between two cored carbons.

At frequencies above 90,000 the power-factor is + 1, therefore the excursions of the P.D. and current obey OHM's law, and the impedance of the arc is equal to its true resistance. So that *the true resistance of an arc,* 3 *millims. long, between* 11 *millims. solid "Conradty Noris" carbons, and through which a current of* 9·91 *ampères is flowing, is* 3·81 *ohms.*

The P.D. between the terminals of the arc, accounted for by ohmic drop in the arc, is therefore 37·8 volts out of an observed P.D. arc of 49·8 volts, so that *there appears to be a real back electromotive force opposing the flow of the currents in this arc of* 12 *volts.*

Considering next Curve II. for both *cored* carbons, the power-factor at the lowest frequency of 250 ∼ per second has a positive value of + 0·67 and increases asymptotically, as in the case of solid carbons, until it is practically + 1 at a frequency of 15,000, and remains unity within the limits of experimental errors up to the highest frequency tried of 50,000 ∼ per second, the impedance becoming practically constant, as with solid carbons.

Therefore *the true resistance of an arc* 3 *millims. long, between* 11 *millims. cored "Conradty Noris" carbons, and through which a current of* 10 *ampères is flowing, is* 2·54 *ohms, and the back E.M.F. is* 16·9 *volts*, calculated in the same way as for solid carbons.

Finally, therefore, arcs between either solid or cored carbons have both back E.M.F. and resistance, and the true values of these quantities differ greatly from those usually assigned to them.

In order to test whether the R.M.S. value of the added alternating current affected the values obtained for the impedance and power-factor, the testing current was varied over the range 0·036 ampère to 0·130 ampère, and the impedance and power-factor were found to be constant within the limits of experimental error.

It is of interest to enquire how the results obtained by Messrs. FRITH and RODGERS can be explained by the aid of these curves. The quantity that they measured and called the resistance of the arc was really the impedance of the arc and a certain resistance together, less the impedance of the resistance part alone. From fig. 2, p. 311, using the same notation as before and assuming R non-inductive,

$$\mathrm{PVC} = \mathrm{V_RC} + \mathrm{P_AV_AC}, \quad \text{or} \quad \mathrm{PV/C} = \mathrm{R} + \mathrm{P_AI_A}.$$

The quantity measured by Messrs. FRITH and RODGERS was $\mathrm{V/C} - \mathrm{R}$.

In the case of *solid* carbons at their frequencies, about 100 ∼ per second, $\mathrm{P_A}$ is roughly − 1, and a little consideration shows that under these conditions P was practically + 1, so that the quantity they called the resistance of the solid arc was $\mathrm{P_AI_A}$, or the product of the power-factor into the impedance of the arc, which is evidently a negative quantity for frequencies under 1950 ∼ per second with the solid arc investigated in Curves I. As they did not use the same make or size of carbons as those used in this paper, it is impossible to make an accurate comparison between the results obtained. Taking arcs of the same length and current between "Apostle," "Brush" and "Carré" carbons, the mean of the value of what Messrs. FRITH and RODGERS called the resistance of the arc is about 0·79 ohm, and this agrees very well with the value of P_AI_A obtained by extra-polating Curves I. back to about 100 ~ per second. So that this curve explains both the sign and the value of the so-called negative resistance of the solid arc.

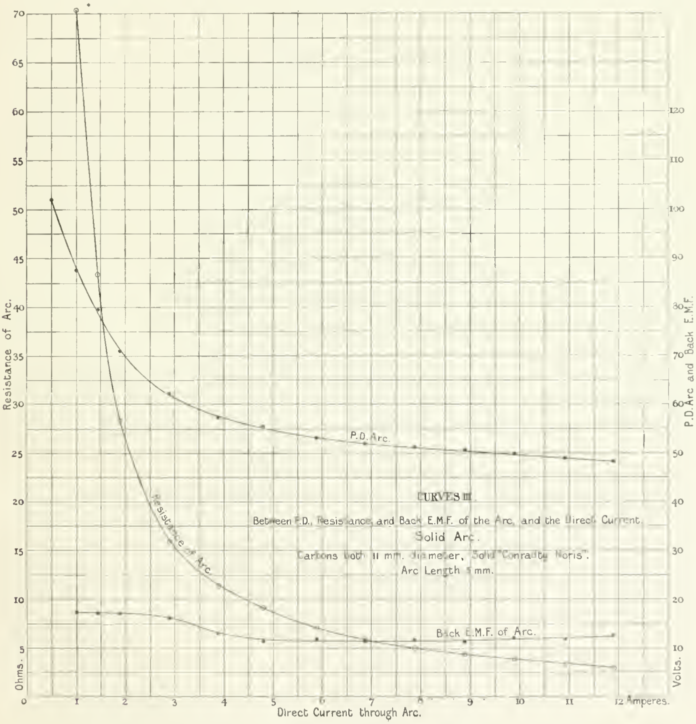
With *cored* carbons, P_A will only be about + 0·5 at 100 frequency. The value of P is unknown, but is certainly less than unity, so that the quantity measured by Messrs. FRITH and RODGERS for cored carbons must be less than P_AI_A. By extrapolation from Curves X., the value of P_AI_A at 100 frequency for cored "Conradty Noris" carbons is about + 0·5 ohm. Owing to the indefinite nature of cored carbons, this value cannot be expected to agree very well with those obtained by Messrs. FRITH and RODGERS for carbons of other makers. Taking arcs of the same length, and current as that used in Curves II., the values given by Messrs. FRITH and RODGERS for different makes are:—"Apostle," 0·03 ohm; "Brush," 0·59 ohm; "Carré," 0·55 ohm. The two latter values agree with the value 0·5 of P_AI_A deduced from the curves for "Conradty Noris" carbons to a higher degree of accuracy than might reasonably be expected, whereas the disagreement in the case of "Apostle" carbons, as measured by Messrs. FRITH and RODGERS, seems to indicate that their result is in some way abnormal, possibly due to some accidental impurity in the core.

The critical frequency of 1·8 ~ per second observed by MESSRS. FRITH and RODGERS for the cored arcs, at which the quantity they measured changes sign, evidently corresponds with the power-factor changing sign, so that the curve between power factor and frequency would cut the zero line at a frequency about 1·8 per second, and it is evident that at lower frequencies it would become negative, an increase of current being accompanied by a decrease in P.D. The curves for the cored arc, if they could be obtained down to sufficiently low frequencies, would therefore be similar to those obtained for the solid arc. The great difference between the solid and cored arcs lies in the much greater quickness with which the conditions of the former can vary, corresponding to any change, however small, in the current through the arc. The relative sluggishness of the cored arc is probably due to the presence in its vapour column of saline matter derived from the core.

The fact that the solid arc has a negative power-factor at frequencies below the critical frequency of 1950 ~ indicates that the arc is under these conditions supplying power to the alternating current circuit, and that this is the fact can easily be shown experimentally by connecting a wattmeter so as to measure the power supplied to the solid arc by the alternating current, when it will be found that at low frequencies *the solid arc is actually supplying power to the alternate-current circuit, while at frequencies above the critical value the alternate-current circuit supplies power to the arc.* This observation is of course not in any way at variance with the principle of conservation of energy, since the alternating energy given out by the arc is derived from the direct-current energy supplied to it, the arc acting as a converter. This fact, that the solid arc is capable, under suitable conditions, of automatically transforming energy derived from a source of direct current into alternate-current energy of any frequency over a wide range, is the explanation of the transformation of energy observed in the Musical Arc recently shown for the first time at the Institution of Electrical Engineers.*

Effect of Varying the Direct Current.

Having found that with a sufficiently high frequency it was possible to measure the true resistance of the arc, this resistance was measured for arcs of a fixed length



CURVES III.

Between P.D., Resistance, and Back E.M.F. of the Arc, and the Direct Current.

Solid Arc.

Carbons both 11 mm. diameter, Solid "Conradty Noris".

Arc Length 5 mm.

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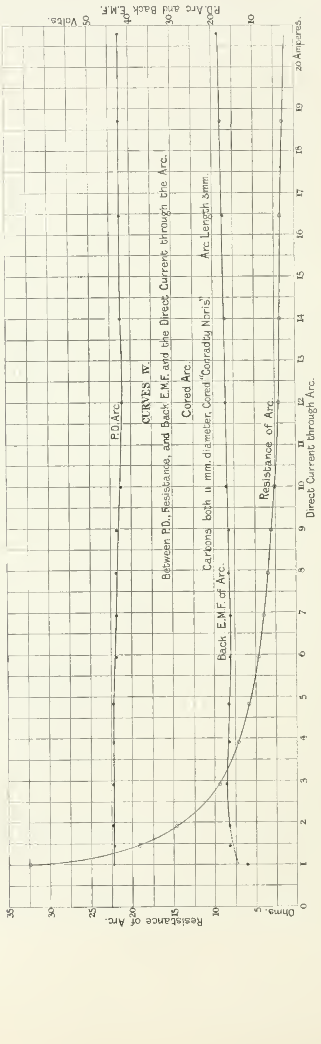
* 'Journal of the Institution of Electrical Engineers,' 1901, vol. 30, p. 248.

of 3 millims. with different values of the direct current. The results of these tests for solid and cored arcs are plotted in Curves III. and IV. The frequency of the



testing current was 100,000 ∼ per second with solid arcs, and 26,000 ∼ with cored arcs. In every experiment the power-factor of the arc was also determined, and found equal to + 1 to within the limits of experimental error, except for a very few small current arcs whose resistance was so high that the power-factor could not be measured with any certainty.

In the Curves III. and IV. there is plotted, besides the resistance of the arc and the P.D. between its terminals, the back E.M.F., calculated as the difference between the P.D. and the product of the direct current into the resistance, or ohmic drop in the arc.

The resistance of both the cored and the solid arcs increases with decrease of the direct current, apparently tending to become infinite for current zero. The back E.M.F. of the solid arc first decreases with increase of current and then slightly increases again, having a minimum of 11·3 volts at about 6 ampères. With cored carbons the back E.M.F. increases with increase of current, from 12·2 volts at 1 ampère to 18·5 volts at 20·8 ampères. It is curious to note that the back E.M.F. of *solid* arcs is larger than that of *cored* arcs for small currents, the reverse being the case with the larger currents.

As the back E.M.F. does not vary much for any of the arcs, the whole of the values observed being between 11·2 volts and 18·5 volts, the high P.D.'s required to maintain very small-current solid arcs is mainly due to the resistance of the arc, and not to the change in its back E.M.F.

The connection between the resistance r and the current A for the *cored* arc, length 3 millims., between 11-millim. "Conradty Noris" carbons, can be approximately expressed over the range from 1·5 ampères to 20 ampères by the very simple relation $(r + 0{\cdot}25)\mathrm{A} = 29$.

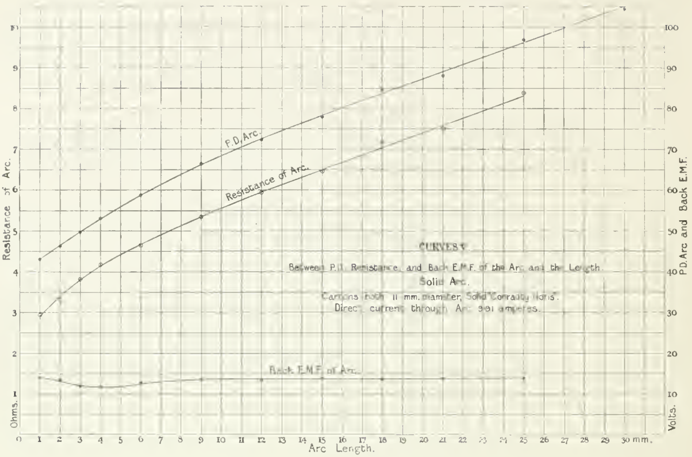
For the *solid* arc, length 3 millims., between 11-millim. "Conradty Noris" carbons, no such simple relation seems to exist; but the curve may be approximately represented over the range 1·5 ampères to 11 ampères by the relation $r = 33{\cdot}5\mathrm{A}^{-1} + 42\mathrm{A}^{-2}$.

Effect of Varying the Arc Length.

The direct current through the arc being kept constant, the connection between the back E.M.F., the resistance, and the length, is given in Curves V. and VI.

With both *solid* and *cored* arcs the effect of increasing the length is to increase the resistance, though not proportionately to the length, the curve between resistance and length being very similar to that between P.D. arc and length. This latter curve is generally assumed to be a straight line, but such is not the case over the wide range of lengths 1 millim. to 30 millims. used in these experiments (see Appendix I.).

The back E.M.F. of the *solid* arc is nearly independent of the length, dropping slightly to a minimum at 4 millims. and then rising again. With the *cored* arc the back E.M.F. decreases with increase of length.

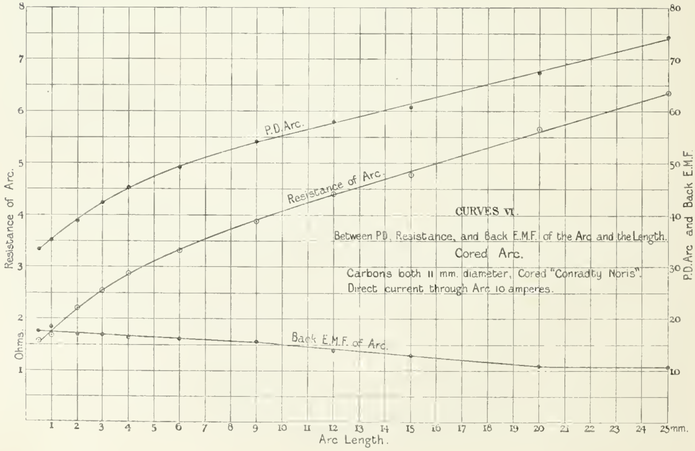

CURVES V.

Between P.D., Resistance, and Back E.M.F. of the Arc and the Length.

Solid Arc.

Carbons both 11 mm. diameter, Solid "Conradty Noris".

Direct current through Arc 9·91 amperes.


CURVES VI.

Between P.D., Resistance, and Back E.M.F. of the Arc and the Length.

Cored Arc.

Carbons both 11 mm. diameter, Cored "Conradty Noris".

Direct current through Arc 10 amperes.

Effect of Varying the Nature of the Electrodes.

The results of experiments to determine whether the resistance and back E.M.F. depend on the size and nature of the electrodes are given in Table IV. The effect of the size of the electrodes is not well marked in the case of "Conradty" and "Apostle" carbons, the changes in size being probably too small to make the effect very evident; the resistances of the arcs are, however, slightly larger with the smaller carbons. With "Le Carbone" Electrographitic solid carbons the impedance of the arc between two 11-millim. carbons is about 17 per cent. higher than that between two 9-millim. carbons. Observations on the arc between these two 11-millim. carbons, which was very unsteady and difficult to maintain, lead the author to think that the observed difference in impedance is not due to the change in size of the electrodes, but to the material of the two sizes of carbons being different, though nominally the same. It is also to be noted that the power-factor, 0·92 of the arc between the 11-millim. carbons, is the only one which has not been found equal to + 1, to within the limits of experimental error, at a frequency of 100,000 ⁓ per second. To be quite certain that this was not owing to some error the experiment was repeated, but with practically identical results.

It may be mentioned that these Electrographitic carbons are not ordinary arc lamp carbons, but were specially made for the experiments by "Le Carbone." They are supposed to consist of pure graphite, and they are said to be made by expelling the remaining impurities from carefully prepared carbon by heating it in an electric furnace until the impurities are volatilised.

Both the resistance and the back E.M.F. of the arc depend greatly on the make of carbon, that is to say, on the composition of the electrodes, since it is very improbable that any two makers' carbons have identical chemical composition. The experiment of soaking a pair of solid "Conradty Noris" carbons in potassium carbonate, drying, and re-determining the resistance, shows that the effect of introducing this potassium salt was to reduce the resistance from 3·81 ohms to 2·92 ohms, and to increase the back E.M.F. from 12 volts to 15 volts for the same arc length and current. A similar effect is produced by drilling out one of the carbons and inserting a glass rod as a core, probably due to the introduction of sodium into the vapour column. The lower resistance and higher back E.M.F. of arcs between cored carbons than of those between solid carbons is also probably due to a similar cause, namely, the presence of potassium silicate in the core. In fact, it seems probable that the whole of the observed differences between solid and cored arcs, and between arcs for which different makes of carbons are used, not only in resistance and back E.M.F., but also in all their physical properties, are due to the different amounts of the traces of foreign substances present in the arc.

The author believes that if it were possible to obtain *perfectly pure carbon* electrodes, then the resistance of the arc between them would be very high, so high that it might be impossible to maintain a true arc between them at all. He is of the opinion that traces of impurities, such as the vapours of the alkaline earths, are essential to provide the carriers of the electric charges in the vapour column, so as to render it conducting and the electric arc as we know it a physical possibility. Unfortunately it has not, up to the present, been possible to obtain pure carbon electrodes in order to test this theory. In favour of it is, however, the known fact that, given an arc of fixed length and current between the best commercial solid carbons, then any addition to it of such substances as potassium or sodium reduces the P.D. required to maintain the arc and its resistance and increases its stability.

The difference between the 11-millim. and 9-millim. Electrographitic carbons mentioned above is probably caused by the last traces of impurities having been more completely expelled in the manufacture from the 11-millim. size than from the 9-millim.

TABLE IV.—Various Carbon Electrodes.

Arc length 3 millims. Added alternating currents 0·1 ampère
Direct current through arc 9·91 ampères. Frequency 100,000 ∼ per second.

Nature of Electrodes Varied.

Make and description of carbon electrodes used.	Diameter.		Direct P.D. arc.	Resistance of arc.	Power-factor of arc.	Resistance of arc × current.	Back E.M.F. of arc.	Remarks.
	+	−						
	millims	millims.	volts.	ohms.		volts.	volts.	
“Conradty Noris” solid .	11	11	49·8	3·81	$0\cdot 99_2$	37·8	12·0	
” ” ” .	11	9	49·8	3·83	$0\cdot 99_9$	38·0	11·8	
” ” ” .	9	9	50·8	3·90	$0\cdot 99_1$	38·6	12·2	
“Apostle” solid . . .	11	11	49·3	4·05	1·00	40·1	9·2	
” ” . . .	11	9	49·9	4·07	$0\cdot 99_7$	40·3	9·6	
“Brush” solid	11	11	50·6	4·04	$0\cdot 99_1$	40·0	10·6	
“Le Carbone” solid . .	11	11	50·4	4·26	1·00	42·2	8·2	
“Le Carbone” . . .	11	11	51·5	4·66	$0\cdot 91_7$	46·1*	5·4	Very unsteady arc.
Electrographitic. . .	11	9	51·2	4·45	$0\cdot 99_1$	44·1	7·1	Fairly steady arc.
Solid	9	9	50·1	3·95	1·00	39·1	11·0	Very steady arc.
“Conradty Noris” solid, soaked in 10 per cent. solution of K_2CO_3 for 36 hours and dried .	11	11	44·5	2·92	1·0	28·9	15·6	Potassium rapidly burnt out of carbons, results are means of two sets of readings only.
“Conradty Noris” solid, centre of negative drilled out and filled up with a glass rod 2·5 mlliims. diameter	11	11	33·1	2·08	$0\cdot 98_5$	20·6	12·5	Very unsteady arc, results are means of 6 sets of readings.
Arc replaced by 3·5 ohm resistance, described page 317	—	—		3·50	$0\cdot 99_9$	—	—	

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* This is an impedance, as the power-factor is not unity, the only one not found to be unity within the limits of experimental error.

Seat of the Back E.M.F. Search Carbons in the Arc.

The fact that the arc has a back E.M.F. which appears to increase with the amount of foreign substances present in the vapour column, at once leads to the question whether this E.M.F. is located at one or the other of the electrodes, or distributed along the vapour column. In order to obtain an answer to this question, some experiments were made on a 6-millim. 9·91 ampère *solid* arc by introducing a search carbon, 2 millims. diameter, into the arc, and measuring not only the direct P.D. between the search carbon and each of the main carbons, but also the impedance to the high-frequency testing current of that part of the arc between it and each ot the main carbons.

In the experiments, three different positions of the search carbon were employed, (1) with its centre 1 millim. from the positive electrode, (2) central in the arc, (3) with its centre 1 millim. from the negative electrode. The fine point to which the search carbon burns was always kept, so far as possible, just reaching to the axis of the main carbons. The results of these experiments are given in Table V.

The introduction of a search carbon into an arc always greatly disturbs the conditions of the arc, and the present case was no exception. The introduction of the search carbon increased the direct P.D. arc by 4·0 volts, and the impedance of the arc lamp by 0·44 ohm. So that the introduction of the search carbon, either by deflecting the arc and so increasing its length, or by chilling the vapour column, increases its resistance by an amount which approximately accounts for the observed increase in P.D. arc. The back E.M.F. of the arc, as a whole, was but little affected by the introduction of the search carbon. This distortion of the arc by the search carbon probably also accounts for the observation that the measured impedance of the arc as a whole is not equal to the sum of the impedances of the two parts comprised between the search carbon and the main electrodes.

Owing to the correct method of apportioning between the two electrodes, the resistance and self-induction of the loop formed by the carbons, holders, and frame ot the lamp, being unknown; and owing to the fact that the measured quantities are only roughly approximate, due to the disturbing effect of the search carbon, no attempt was made to apply the small correction to the observations for the self-induction and resistance of the carbon holders and lamp frame, and the observed impedances were treated as resistances, and the back E.M.F.'s calculated as usual. Further, the three arcs which had the same length and current will be considered as having been identical, though such was not strictly the case.

On these assumptions, consider the resistance between the positive electrode and the search carbon when the search carbon is 1 millim. from the positive electrode, and then 5 millims. from the positive electrode (*i.e.*, 1 millim. from the negative). The change in resistance due to this change of 4 millims. in the position of the electrode is 1·72 ohms. Taking next the measurements made between the negative electrode and the search carbon, the difference for the same movement of the search carbon is 1·64 ohms. The mean of these two results is 1·68 ohms for a movement of the search carbon of 4 millims. If this distance really represented the length of the vapour column between the two positions of the search carbon, and if its resistance is uniform, then its resistance per millim. would be 0·42 ohm. It is probably less than this, owing to the length of the vapour column between the two positions being appreciably longer than 4 millims., due to its distorted shape

TABLE V.—Search Carbon in Arc.

Carbons both 11 millims. diameter. Solid "Conradty Noris."
Arc length 6 millims. Direct current through arc 9·91 ampères.
Added alternating current 0·1 ampère. Frequency 100,000 ~ per second.

Position of search carbon.	Direct P.D. arc lamp.	Direct P.D. between search carbon and + electrode.	Direct P.D. between search carbon and − electrode.	Impedance of arc lamp.	Impedance between search carbon and + electrode.	Impedance between search carbon and − electrode.	Impedance of arc lamp × direct current.	Impedance between search carbon and + electrode × direct current.	Impedance between search carbon and − electrode × direct current.	Back E.M.F. of arc lamp.	Back E.M.F. between search carbon and + electrode.	Back E.M.F. between search carbon and − electrode.
	volts.	volts.	volts.	ohms.	ohms.	ohms.	volts.	volts.	volts.	volts.	volts.	volts.
One millim. from + electrode .	64·3	37·8	25·3	$5\cdot3_1$	$2\cdot1_4$	$3\cdot2_1$	$52\cdot{}_6$	$21\cdot{}_2$	$31\cdot{}_8$	$11\cdot{}_7$	$16\cdot{}_6$	$6\cdot{}_5$
Central . . .	62·8	43·9	18·9	$5\cdot0_5$	$2\cdot6_2$	$2\cdot5_5$	$50\cdot{}_0$	$26\cdot{}_0$	$25\cdot{}_3$	$12\cdot{}_8$	$17\cdot{}_9$	$6\cdot{}_4$
One millim. from − electrode .	63·8	53·9	10·3	$5\cdot1_8$	$3\cdot8_6$	$1\cdot5_7$	$51\cdot{}_3$	$38\cdot{}_3$	$15\cdot{}_6$	$12\cdot{}_5$	$15\cdot{}_6$	$5\cdot{}_3$
Mean values .	63·6	—	—	$5\cdot1_8$	—	—	$51\cdot{}_3$	—	—	$12\cdot{}_3$	$16\cdot{}_7$	$6\cdot{}_1$

Assuming the resistance to average 0·42 ohm per millim., then the resistance of the 6 millims. of *vapour column* is only 2·5 *ohms*, as against 5·2, the measured value for the lamp and arc. There must therefore be a large resistance at or near the electrodes. Calculating its value from each of the three experiments, the resistance at or near the *positive* electrode is 1·72, 1·36 and 1·76 ohms respectively, mean 1·61 *ohms*; and the resistance at or near the *negative* electrode is 1·11, 1·29 and 1·15 ohms respectively, mean 1·18 *ohms*. As there seems no very good reason to suppose that the resistance of the vapour column is very much greater near the electrodes than elsewhere, the above resistances are probably located at the contact between the electrodes and the vapour column.

The resistance of the arc as a whole may therefore be considered to consist of three parts, the resistance of the vapour column and the resistances of the two contacts between it and each of the electrodes.

The mean value of the back E.M.F. of the arc lamp, with the search carbon in place in the arc, *i.e.*, 12·3 volts, is practically identical with its value of 12·7 volts, obtained without any search carbon in the arc. The back E.M.F. of that part of the arc between the search carbon and the positive electrode had, for the three different positions of the search carbons, the values 16·6, 17·9 and 15·6 volts respectively, which may be considered as indicating that this back E.M.F. is independent of the position of the search carbon to within the limits of accuracy of the present experiments. The back E.M.F. between the search carbon and the negative electrode calculated in the same way, *i.e.*, by subtracting the product of resistance by direct current, or ohmic drop in this part of the arc, from the observed direct P.D. between the search carbon and the negative electrode, is a *negative quantity*, so that there exists a *forward* E.M.F. which helps the flow of the direct current. The value of this forward E.M.F. was 6·5, 6·4 and 5·3 volts respectively, again practically independent of the position of the search carbon.

These experiments show that there is in this arc at or near the positive electrode a *back* E.M.F. having a mean value of 16·7 volts, and at or near the negative electrode a *forward* E.M.F. of 6·1 volts. The fact that the algebraic sum of these two voltages is not equal to the observed mean back E.M.F. 12·3 volts of the arc, as a whole, is probably caused by the distortion of the lines of flow of the current through the arc, produced by the introduction of the search carbon, which renders the sum of the resistances measured between each of the main electrodes and the search carbon greater than their total as measured between the main electrodes.

The back E.M.F. of the arc as a whole consists, therefore, of two E.M.F.'s located at or near the electrodes, the larger E.M.F. situated at or near the positive electrode opposing the flow of the current, the smaller E.M.F. situated at or near the negative electrode helping the flow of the current.

The approximate values of the resistance E.M.F. of each part of the arc, and the drop of volts due to each of these causes, corrections for the self-induction and resistance of the holders and carbons being omitted as before, are given in Table VI. The P.D.'s in this table multiplied by the direct current of 9·91 ampères give the power supplied to each part of the arc.

TABLE VI.—Distribution of the P.D. in a Solid Arc.

Carbons, both 11-millims. "Conradty Noris." Arc length 6 millims. Direct current through Arc 9·91 ampères.

	Ohmic drop due to resistance.	Back E.M.F.	Sum of back E.M.F. and ohmic drop.
	volts.	volts.	volts.
At or near the + crater	+16·0	+16·7	+32·7
Vapour column	+25·0	0	+25·0
At or near the crater	+11·7	− 6·1	+ 5·6
Total	+52·7	+10·6	+63·3

Conclusion.

Ever since the arc was first assumed to have a back E.M.F., speculation has been rife as to its cause and its location in the arc. So long as the value assigned to it was of the order of 40 volts, and it was supposed to be located only at the crater surface, there was great difficulty in offering any consistent explanation of it. It remains, therefore, to consider whether the new facts set forth in this paper render the matter more susceptible of a satisfactory explanation.

So far as the resistance of the arc is concerned, there seem to be no difficulties, the known relations between the size and shape of the vapour column, the size of the craters, the current, and the arc length. explain the observed changes in resistance when the two latter variables are altered. The magnitude of the resistance of the vapour column and of the contacts between it and the electrodes are not such as to offer serious difficulties, nor does the fact that they are altered by the presence of foreign bodies.

Any explanation of the back E.M.F. of the arc as a whole must, in the light of the results given in the last section, account for the existence of two unequal E.M.F.'s of opposite signs, of which the larger opposes the flow of the current. The observed back E.M.F. of the arc is the resultant of these two, their values being of the order of 17 volts and 6 volts respectively in the solid arc. The existence of two E.M.F.'s of opposite signs, situated at or near the electrodes, considerably simplifies matters, since any explanation which would account for a back E.M.F. in the direction carbon-to-vapour would probably also explain a forward E.M.F. vapour-to-carbon. These E.M.F.'s are probably either due to a polarisation at the electrodes or to thermo-electric forces. The polarisation E.M.F.'s include those due to chemical changes and those which have been assumed to be caused either by the volatilisation or the tearing-off of particles from the electrodes. There appears to be very little evidence in favour of these last two explanations.

The statement sometimes made that, as it requires a certain amount of work to be done to convert the solid electrodes into the gaseous state, and as this work is done by the current, therefore the current must be flowing against a back E.M.F., is without sufficient foundation. It is true that, in the above case, a P.D. will exist which opposes the flow of the current, but that this P.D. is a reversible phenomenon, and therefore an E.M.F., is not necessarily the case. The volatilization of the electrode is of course reversible, but it requires experimental evidence to prove that it is accompanied by a corresponding reversible electric phenomenon, so that the energy, supposed supplied electrically, to cause the volatilisation, tends to be returned, on condensation, in the form of electric energy. There is no evidence that any appreciable E.M.F. is produced by the tearing-off of the solid particles from the electrodes, and HERZFELD's experiment of attracting these particles out of the arc by means of an electrostatic field, which he says did not affect the P.D. or current through the arc, seems to indicate that the back E.M.F. of the arc cannot be due to this cause.

That more than a small part of the back E.M.F. is due to a polarisation, such as occurs in an ordinary cell, is difficult to conceive. If such were the case, what is the nature of the chemical compound produced, and what becomes of it? It is certain that practically the whole of the energy supplied to the arc is emitted again as light and heat, and that there is no considerable portion stored up in the products produced by the arc. If, therefore, any chemical change takes place at the positive electrode accompanied by absorption of energy, this energy must be given out again in the arc or flame, and the reverse chemical change take place. Some of the energy might be given out at the negative electrode, and account for the forward E.M.F. observed there. The nature of the substance in which the chemical change must take place, which change reverses and so forms a cyclic process, is an almost insuperable difficulty, since the rate at which the energy must be constantly absorbed and given out again is considerable, and the materials present in which this chemical change must take place are only carbon, its vapour, and the slight trace of impurities which the author thinks essential to the existence of the arc.

If the impurity be assumed to be a salt of potassium, there is the possibility that a carbide of potassium might be formed, and that part of the P.D. might be due to the arc forming a cell, having carbon and potassium carbide as its electrodes, and the vapour column as the electrolyte, the products produced by the flow of the current through the cell being conceived to be destroyed in the flame which surrounds the arc proper.

That a chemical combination between the carbon and surrounding gas is not the cause of the back E.M.F., at any rate in the normal silent arcs considered here, is evident from the fact that the arc is but little affected by the nature of the gas in which it burns, as has been shown by Professor S. P. THOMPSON and others. Putting aside the possible combination of the carbon with the slight trace of impurity, which may account for a small part of the back E.M.F., the polarisation E.M.F.'s do not seem able to account satisfactorily for the whole back E.M.F. of the arc, without making assumptions which are at present unsupported by any satisfactory evidence.

Against the back E.M.F. of the arc being due to thermo-electric forces, it is generally urged that these forces are usually reckoned in tenths and hundredths of a volt, and that therefore the order of magnitude of the back E.M.F., which was then supposed to be about 40 volts, rendered it highly improbable that it was due to thermal causes.

There is no doubt that the temperature of the positive crater is very high—said to be about 3500° C., and it is quite possible that there may exist large temperature gradients near the electrodes, since the true back E.M.F. to be accounted for is only of the order of 17 volts at the positive crater. It is therefore worth while re-considering whether after all the back E.M.F. of the arc may not be due to thermo-electric forces.

In this connection the experiment of DUBS[*] is of considerable interest. He took two carbon plates, about 1 millim. apart, and caused a blow-pipe flame to impinge across their edges, and found that a small current could be obtained from one plate to the other through a galvanometer. He considered this result might be analogous to the back E.M.F. of the arc. A similar experiment has also been described by OLIVETTI.[†]

The views of the author that impurities, such as the salts of the metals of the alkaline earths, are essential to the existence of the arc, have led him to try a considerable number of experiments on the P.D.'s produced by unequally heating carbon electrodes, either with the addition of such salts to the electrodes, or to the flame used as a source of heat. Most of these experiments were carried out by heating the tips of ordinary arc-lamp carbons held in the hand-feed arc lamp already described, by causing a "blow-through" or a "mixed" oxy-coal-gas flame to impinge upon them.

The terminals of the arc lamp were connected to a direct-current voltmeter and were not joined in any way to a source of current, so that the P.D.'s observed were not due to leakage from any extraneous source. The choice of the voltmeter presents some difficulty. At first sight, owing to the high resistance of the flame between the two tips of the carbons, it might seem that an electrostatic or a very high resistance voltmeter would be the best. Electrostatic voltmeters were however found unreliable, owing to the friction of the gases in the jet and against the carbons charging them up electrostatically. Two voltmeters were therefore used, the one an ordinary Weston pivot instrument, having a resistance of about 600 ohms, and the second a reflecting moving-coil instrument of 20 ohms, having a resistance of about 16,000 ohms in series with it. The resistance of both these instruments is so low that no deflection due to the frictional charges could be observed on them. By comparing the P.D.'s obtained when one or the other of the voltmeters was used an estimate of the resistance of the vapour between the carbon tips could be made.

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[*] 'Beiblätter,' 1889, vol. 13, p. 197.

[†] 'Electrical Review,' 1892, vol. 31, p. 728.

If the flame was caused to impinge on the two carbon tips so as to heat them equally, as judged by eye, then no P.D. was observed between them. If the flame was now moved so as to heat one carbon more than the other, then a P.D. was observed between the carbons, the hotter being positive to the cooler as indicated by the voltmeter, that is, in the same direction as the back E.M.F. of the arc, assuming that the positive crater is the hotter.

The highest P.D. obtained was 1·5 volts when using cored carbons which had been previously soaked in potassium carbonate solution. This P.D. was not in any way due to differences in the quality of the two carbons, since by moving the jet so as to heat either carbon more than the other the P.D. changed sign, the hotter carbon always being positive. By setting one carbon in front of the other it was found that the direction of the P.D. was unaffected by whether the stream of flame gases flowed from the hotter to the cooler carbon or *vice versâ.*

With two pieces of the same solid carbon, about 2 millims. apart, no foreign bodies being introduced in any way, the highest P.D. obtained was about ·0·5 volt on the higher resistance voltmeter, and the resistance of the flame between the carbons was deduced as about 4000 ohms, so that the E.M.F. between these *solid* carbons was only about 0·62 volt. After soaking this same pair of carbons in potassium carbonate the P.D. obtained was 1·0 volt., the resistance of the vapour being only a few hundred ohms, so that the effect of introducing this potassium salt was to greatly increase the E.M.F. between the carbons. If these E.M.F.'s are due to Peltier effects, then it would seem as if the introduction of the potassium greatly increased the thermo-electric power of the junction.

The above experiment is in agreement with the fact that the back E.M.F. of the cored arc has been generally found larger than that of the solid arc, as the cored arc undoubtedly contains more impurities.

By varying the proportions in which the oxygen and coal gas were mixed before burning, it was found that the highest P.D.'s appeared to correspond with the gases being burnt in their combining proportions, so as to produce the hottest flame; if either gas was considerably in excess, a much lower P.D. was obtained. It is to be noted, however, that by suppressing the coal gas altogether and allowing the carbon to burn in the oxygen, P.D.'s up to 1·5 volt could be obtained, as before the hotter electrode always being positive to the voltmeter. These latter experiments, though not conclusive, are not in favour of the P.D. being due directly to a combination of the carbon and either of the gases.

The temperature of the positive crater of the arc is much higher than the highest temperature obtained by the oxy-coal-gas flame, and the temperature differences and gradients which may exist in the arc are probably many times greater than those obtained in any of the above experiments, for it must be remembered that the carbon at the lower temperature must always be at a bright red heat, or else it does not appear to make electrical contact with the flame. It does not therefore seem improbable that the P.D.'s of 1 volt or 1·5 volts obtained by unequally heating the two carbons may have the same origin as the 10-volts to 18-volts back E.M.F. found in the arc, especially as they agree both in direction and in the effect of impurities on them. On this assumption, the probable causes of the back E.M.F. of the arc reduce themselves to two, viz. :—

(1.) A thermo-electric force at the junction of the carbon and vapour, causing the major part of the observed back E.M.F.

(2.) A combination of carbon with the impurity present.

Whether the thermo-electric force at the junction of carbon and vapour be due either to the Peltier effect, or to a high temperature gradient at the contact, *i.e.*, a Thomson effect, or both, it seems at the present time to afford the most satisfactory explanation of the back E.M.F. of the arc and the P.D.'s observed when two carbon electrodes are unequally heated. The main objection is the great difference of thermo-electric power the components of the junction must have, which, if the difference of temperature be assumed to be only 1000° C., amounts to about 15 to 20 times that between bismuth and selenium.

In favour of the view that the back E.M.F. is mainly due to thermal causes may be mentioned the unilateral behaviour of the alternating arcs between a metal ball and a metal point,* between carbons and metals,† and the observation of CROSS and SHEPARD‡ on the effect of cooling the positive crater of the direct-current arc. All these experiments indicate that there is some cause existing which enables the current to flow much more easily up the temperature gradient than down it.

A tentative explanation of the chief causes which oppose the flow of the current through the arc may be given under four heads :—

(1.) The resistance of the true vapour column, which is probably an electrolytic conductor, whose conductivity greatly depends on the traces of impurities, such as potassium and soda, present. Pure carbon vapour, like pure water, has probably a very high specific resistance.

(2.) A high contact resistance between the electrodes and the vapour column, which leads to a large generation of heat, and consequently high temperatures and temperature gradients at these points.

(3.) Possibly a small back E.M.F., due to the electrolytic cell formed by the electrodes and the vapour column as electrolyte.

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* ARCHBOLD and TEEPLE, 'American Journal of Science,' 1891.

† DUDDELL and MARCHANT, 'Journal of the Institution of Electrical Engineers,' 1899, vol. 28, p. 74.

‡ 'American Academy of Sciences,' 1896, p. 227.

(4.) Considerable thermo-electric forces at the contacts between the electrodes and the vapour column, whose resultant is the greater part of the observed back E.M.F. of the arc. The force at the positive opposing the flow of the current, and that at negative helping it, account for the much greater conversion of electric energy into heat at the positive electrode.

In the above, it will be noticed that the volatilisation or sublimation of carbon is not supposed to have any influence on the back E.M.F. except in so far as it limits the temperature the electrodes can obtain.

The introduction of impurities up to a certain limit into the arc will cause the conductivity of the vapour column to increase, and for a fixed current its temperature will probably decrease, so the thermo-forces and the back E.M.F. may be expected to increase as has been observed (see Table IV.).

There is another possible explanation of the larger back E.M.F. when impurities are present, viz., the thermo-electric force of the junction carbon-vapour may be larger the greater the quantity of impurity present. If this is the case, may not the explanation of the drop in P.D., when the arc hisses, or when oxygen or hydrogen come in contact with the positive crater, as found by Mrs. AYRTON, be that *the gas combines with the impurity* and reduces the back E.M.F., and not that it combines with the carbon as suggested by Mrs. AYRTON ?

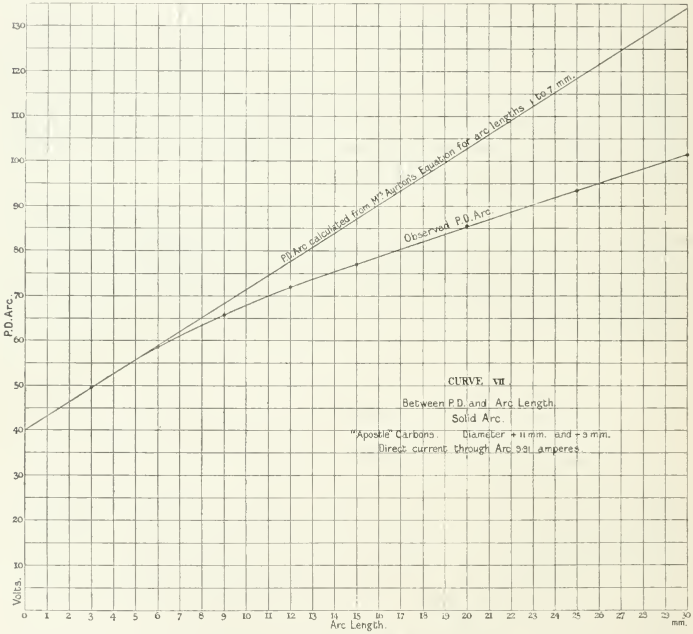
In conclusion, the author wishes to express his thanks to Professor AYRTON and Mr. MATHER of the Central Technical College for the very valuable assistance and advice they have given him during the course of these investigations ; he also wishes to thank the many students who have from time to time helped him with the experiments, and especially Messrs. DEL MARR, LYNN, BROWN, WATSON, and VINES.

APPENDIX I.

On the Relation between the P.D. Current and Length of the Arc.

Mrs. AYRTON has pointed out that the relation between the P.D., the current, and the arc length, may for the solid carbons which she used be accurately expressed by an equation of the form $V = \alpha + \beta l + (\gamma + \delta l)/A$, where V and A are the P.D. and current respectively, and α, β, γ, and δ are constants, between the limits over which she varied the current and length. It was thought, therefore, that it would be of interest to determine the values of the constants α, β, γ, δ for the solid "Conradty Noris" carbons used in so many of the experiments in this paper. The two series of results in Curves III. and V. give P.D. when the length is kept constant and the current varied ; and also the P.D. when the current is kept constant and the length varied; and in order to increase the accuracy of the equation some extra results were obtained in which both current and length have been varied.

On attempting to find the constants α, β, γ, δ to fit all these values, the first difficulty encountered was that, assuming the current to be constant and the length varied, Mrs. AYRTON's equation required the P.D. to be a linear function of the length; a glance at Curve V. shows that this was *not* even approximately true for these carbons and the long range of lengths from 1 millim. to 30 millims. used. At first it was thought that this might in some way be due to the kind of carbons used in the experiments. The connection between P.D. and length for constant current was, therefore, determined for the same size and make of carbons used by Mrs. AYRTON in her experiments*; the results are given in Curve VII.



In the curve the straight line represented by Mrs. AYRTON's equation is also given. The divergence between the P.D.'s calculated from Mrs. AYRTON's equation and the observed P.D.'s is very marked, the error in the calculated result being no less than 32·6 volts for an arc length of 30 millims. At the same time this is not due to any difference in the nature of the carbons, since the actual experimental values obtained by Mrs. AYRTON,* and on which her equation is based, fit the Curve VII. with considerable accuracy. The whole of this large difference between Mrs. AYRTON's equation and the experimental results is due to the fact that the equation is based on the range of arc lengths of from 1 millim. to 7 millims., and that the equation no longer represents the facts if we extrapolate any considerable amount. Even over the range of from 1 millim. to 7 millims. the connection between the P.D. and length for constant current is not accurately a straight line, as can be seen either in Curve VII. or by a careful examination of Mrs. AYRTON's experimental results.

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* The 'Electrician,' 1895, vol. 35, p. 420.

If the arc length was kept constant and the current varied, a similar difficulty was found with very small currents of 1·5 ampères and lower, namely, that the observed P.D. was less than the calculated P.D., so that Mrs. AYRTON's *equation can only be considered as approximately representing the facts within the limits she used, and must not be applied to very small currents or long arc lengths.*

Taking, in the present case, as limits from 1·5 ampères to 12 ampères, and from 1 millim. to 6 millims., then the results obtained with the arc between 11 millims. solid "Conradty Noris" carbons can with a very fair approximation be represented by the equation

$$V = 39{\cdot}6 + 1{\cdot}7l + (15{\cdot}5 + 11{\cdot}5l)/A,$$

which is in very close agreement with the equation given by Mrs. AYRTON for solid "Apostle" carbons, 11 millims. and 9 millims. diameter, for the range 1 millim. to 7 millims., viz.:

$$V = 38{\cdot}88 + 2{\cdot}07l + (11{\cdot}66 + 10{\cdot}54l)/A.$$

APPENDIX II.

On the Resistance of an Electrolyte.

In measuring the resistance of an electrolyte by the ordinary Kohlrausch method, using alternating or induced currents, it is usually assumed that the influence of polarisation of the electrode is avoided, and that the frequency of the alternating current used is unimportant, provided that it is moderately high, say a few hundred periods per second. If an appreciable polarisation of the electrodes is produced by the testing current very soon after its application in any given direction, then the above assumptions will not be true, and the resistance as measured at such low frequencies will differ from that measured at much higher frequencies.

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* 'Electrician,' vol. 35, p. 635.

In order to investigate this matter, the arc was replaced by an electrolytic cell, and its impedance and power-factor tested in a similar way to those of the arc, except that no direct current was sent through the cell. To make the polarisation effects important compared with the ohmic drop, the resistance of the cell must be small, therefore sulphuric acid was chosen as the electrolyte. The cell used was a glass trough 100 millims. long, 104 millims. deep, and having a mean thickness of 15·53 millims. The electrodes were two platinum plates 90 millims. apart (not platinised) which fitted the trough tightly and extended to the bottom.

The mean depth of the electrolyte was 61·6 millims. So that the liquid whose resistance was measured was a rectangular parallelopiped, having a length of 9 centims. and a mean cross-section of 9·57 sq. centims.

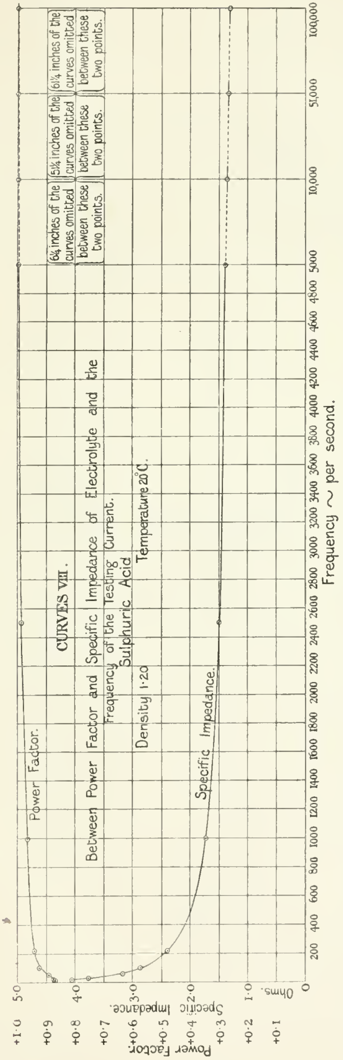
The temperature was kept constant at 20° C. to within 0·1° C. during the test, by immersing the trough in a water-bath. The results of the tests of the impedance and power-factor at different frequencies are given in Curve VIII.

With the above cell, neither the impedance nor the power-factor are independent of the frequency at ordinary frequencies, and it is evident that the polarisation in this electrolyte occurs so rapidly that it is not until the frequency is well above 10,000 ~ per second that it is unable to follow the variations of the testing current, and that the electrolyte behaves like an ordinary resistance. If the resistance of this cell were tested in the ordinary way, at a frequency of about 100 ~ per second, the value obtained would be over twice the true resistance of the cell.

The slight drop in impedance observed when the frequency is increased from 50,000 to 100,000 ~ per second is possibly due to the electrostatic capacity of the plates and liquid in the cell, and to the water-bath surrounding it acting as a shunt to the cell. Attempts were made to determine experimentally the exact value of this correction, which indicated that it was less than 1 per cent. at 100,000 ~ per second.

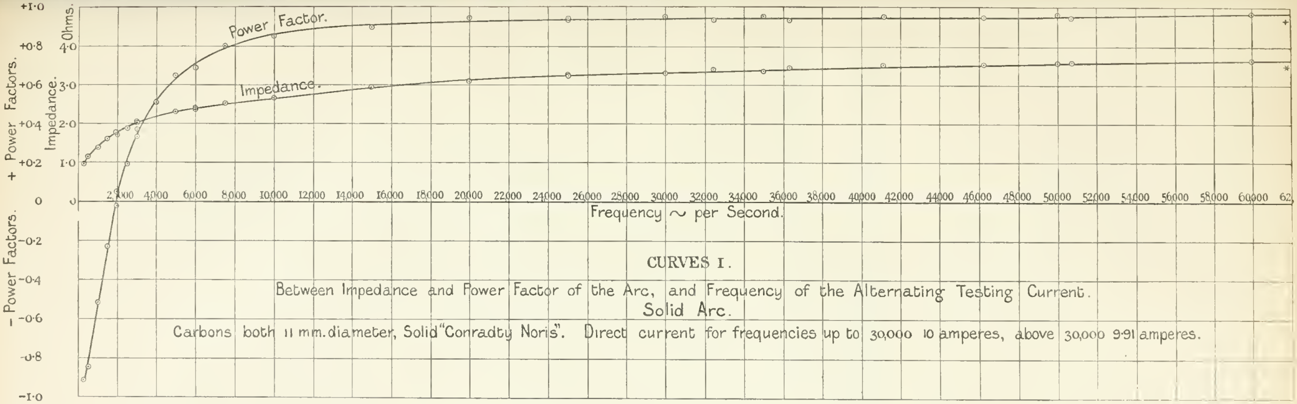
The possible errors due to polarisation, even when using alternating currents, were thoroughly appreciated by Kohlrausch, and in the 'Leitvermögen der Elektrolyte,' by Kohlrausch and Holborn, means are described for minimising these errors. By interpolation from their results the conductivity of sulphuric acid of the same density and at the same temperature as that used in this experiment is 0·758 (ohm × centim.)$^{-1}$, or a specific resistance of $1·319 \times 10^9$ c.g.s., which confirms the value of $1·30 \times 10^9$ obtained in the present experiment at a frequency of 50,000 ~ per second.

The fact that the power-factor is not unity at low frequencies proves that at these frequencies the electrolyte does not behave like an ordinary resistance, and consequently *all* the instantaneous values of the P.D. between the terminals of the telephone sometimes used in the Kohlrausch method cannot be made to vanish at


CURVES VIII.
Between Power Factor and Specific Impedance of Electrolyte and the Frequency of the Testing Current.
Sulphuric Acid
Density 1·20
Temperature 20°C.
Power Factor.
Specific Impedance.
6¼ inches of the curves omitted between these two points.
5¼ inches of the curves omitted between these two points.
61¼ inches of the curves omitted between these two points.
Power Factor.
+1·0
+0·9
+0·8
+0·7
+0·6
+0·5
+0·4
+0·3
+0·2
+0·1
Specific Impedance.
5·0
4·0
3·0
2·0
1·0
Ohms.
0
200
400
600
800
1000
1200
1400
1600
1800
2000
2200
2400
2600
2800
3000
3200
3400
3600
3800
4000
4200
4400
4600
4800
5000
10,000
51,000
100,000
Frequency ~ per second.

these frequencies, even supposing the other arms of the bridge quite free from self-induction and capacity. This explains the fact that only a minimum in the sound is obtainable, and not absolute silence.

The conclusion to be drawn from this experiment is that in any case where the P.D. due to the polarisation of the electrodes cannot be made very small compared with the ohmic drop along the liquid whose resistance is being measured, and where the errors due to the polarisation cannot be eliminated by taking two or more tests, then it *must not be assumed without proof that the use of alternating currents at ordinary frequencies of a few hundred periods per second eliminates the possibility of errors due to polarisation.* For in the case of sulphuric acid used above, the polarisation can vary as rapidly as the resistance of the cored arc.

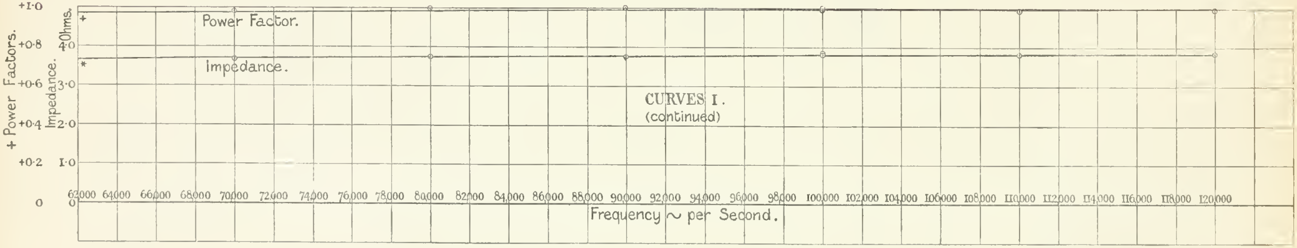


CURVES I.

Between Impedance and Power Factor of the Arc, and Frequency of the Alternating Testing Current.

Solid Arc.

Carbons both 11 mm. diameter, Solid "Conradty Noris". Direct current for frequencies up to 30,000 10 amperes, above 30,000 9·91 amperes.



CURVES I.
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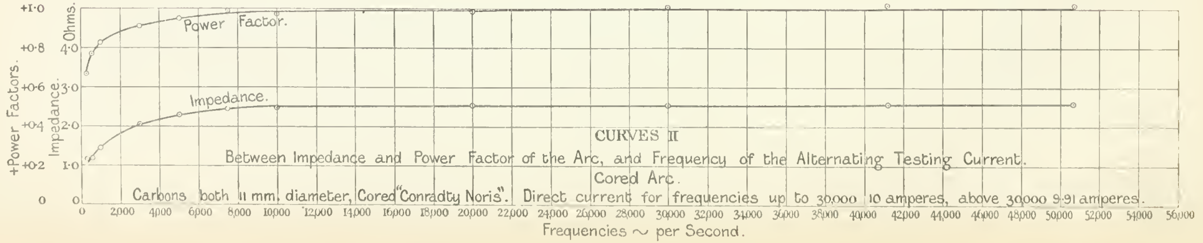

CURVES II

Between Impedance and Power Factor of the Arc, and Frequency of the Alternating Testing Current.

Cored Arc.

Carbons both 11 mm. diameter, Cored "Conradty Noris". Direct current for frequencies up to 30,000 10 amperes, above 30,000 9·91 amperes.

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XI. *On the High-Temperature Standards of the National Physical Laboratory: an Account of a Comparison of Platinum Thermometers and Thermojunctions with the Gas Thermometer.*

By J. A. Harker, *D.Sc., Fellow of Owens College, Manchester, Assistant at the National Physical Laboratory.*

Communicated by R. T. Glazebrook, *F.R.S., from the National Physical Laboratory.*

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I. *Introduction.*

IN a paper "On the Comparison of Gas and Platinum Thermometers," read before the Royal Society in 1900,[*] Dr. P. CHAPPUIS and the author described a series of experiments in which several platinum-resistance thermometers, constructed of wire of specially high purity, were compared with the gas thermometer at a number of steady temperatures from below zero to above the boiling-point of sulphur, and in one set of measurements to just short of 600° C.

The results were such as to substantially confirm the conclusion of CALLENDAR and GRIFFITHS that the indications of platinum thermometers may be reduced to the normal scale by the employment of CALLENDAR's well-known difference formula

$$d \equiv \mathrm{T} - pt = \delta\left[\left(\frac{\mathrm{T}}{100}\right)^2 - \frac{\mathrm{T}}{100}\right]$$

where $d =$ the difference between T, the temperature on the normal scale, and $pt =$ the "platinum" temperature. The constant δ for pure platinum wires is approximately 1·5, the three temperatures chosen for its determination being 0°, 100° and the boiling-point of sulphur.

The paper concludes with the sentence, "until further investigations have been made as to the relations of the various gas scales at high temperatures and as to the influence of the initial pressure and the effect of impurities and traces of water vapour in the gases employed, and until exact determinations have been made up to high temperatures of the coefficient of expansion of the material used as thermometric reservoir, we think that for the purposes of high-range thermometry a scale deduced by the parabolic formula from that of the platinum thermometer will suffice. In the present state of our knowledge any attempt to improve on such a thermometric scale would be attended with such uncertainties as would probably render it futile."

Since that time, however, a substantial advance has been made in our knowledge, direct determinations of the expansion of porcelain up to high temperatures having been made by different observers, namely, Mr. BEDFORD,[†] at Cambridge, and Messrs. HOLBORN and DAY at the Reichsanstalt.[‡] A discussion by Dr. CHAPPUIS of the results obtained by these observers and their influence on high-range thermometry is found in the 'Philosophical Magazine,' (5), October, 1900, and February, 1902.

An examination of the difference formula for the platinum thermometer shows that it can only represent a physical reality over a limited range, the value of pt for a wire having a δ of 1·5 reaching a maximum about 1700° pt, a value numerically not far exceeding such as may safely be attained. It would not therefore be surprising if the formula which actually holds remarkably closely at low ranges should be found to give erroneous results at temperatures well below the maximum to which the materials used in the construction of a platinum thermometer can be subjected without injury. The investigations dealt with in the present paper have been carried out at the National Physical Laboratory during the past two years, and consist mainly of a continuation of the work of CHAPPUIS and the author on the platinum thermometer, testing up to 1000° C. the validity of the difference formula for two thermometers made of representative platinum wires of high purity, by comparison of these instruments with the constant volume gas thermometer. With these instruments were also compared simultaneously standard thermojunctions, whose electromotive force at a series of temperatures had been determined with special care at the Reichsanstalt at Charlottenburg.

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[*] 'Phil. Trans.,' A, vol. 194, pp. 37–134.

[†] BEDFORD, 'Proc. Phys. Soc.,' XVII., Part III., p. 148, and 'Phil. Mag.'

[‡] HOLBORN and DAY, 'Ann. Phys.,' vol. 6, 1901, p. 136.

II. *The Gas Thermometer.*

The gas thermometer employed for this work is a duplicate of the one used by HOLBORN and DAY at the Reichsanstalt. It was obtained from the same maker, FUESS, of Berlin, and was presented to the laboratory by Sir ANDREW NOBLE. For this munificent gift and for the kindly assistance and advice rendered by the President of the Reichsanstalt, Dr. KOHLRAUSCH, and by Dr. HOLBORN in procuring for us the gas thermometer, thermocouple, wire and materials for the construction ot electric furnaces, the laboratory is greatly indebted.

The instrument is specially designed for rapid work at high temperatures, and was arranged so that measurements could be made with any desired initial pressure and with bulbs of different materials. The principle employed by CHAPPUIS, in the two gas thermometers at Sèvres, ot making all the measurements depend upon the determination of a single length, though undoubtedly capable of giving by far the most accurate results, becomes somewhat inconvenient when great changes of pressure are needed. For this reason, therefore, in the present apparatus the manometer is arranged so as to measure directly the difference of height between the level of a very short metal point, to which the mercury in the closed limb A, fig. 1, is adjusted, and the mercury surface in the long tube B, which during the measurements communicates with the atmosphere by the tap H.*

The tubes A and B communicate by means ot cone joints with the lower part of a closed iron reservoir in the base plate of the apparatus, into which mercury can enter from the upper reservoir G by means of the long tube C and steel tap D. The fine adjustment of the height of the mercury to the point in A is made by a steel screw with capstan-shaped head projecting from the bottom of the apparatus and working on a thin steel diaphragm let into the bottom of the reservoir.

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* In the original form of the apparatus, tube A was joined to the reservoir below by a large three-way glass tap, through the side tube of which the filling of the gas into the reservoir was made. It was found, however, that this tap was a source of danger in the measurements, the results of one set of comparisons

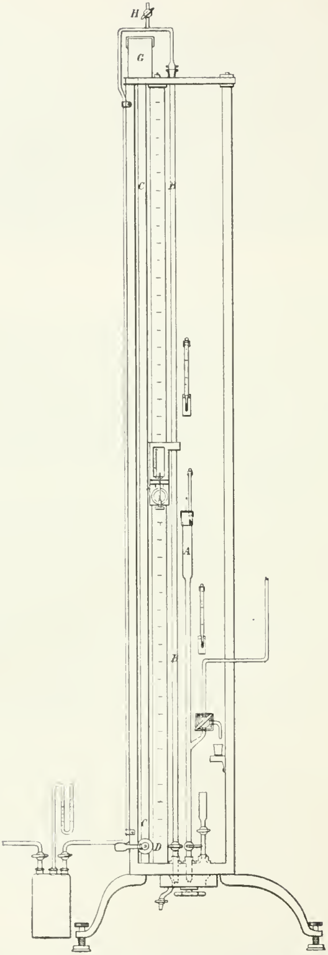

Fig. 1.

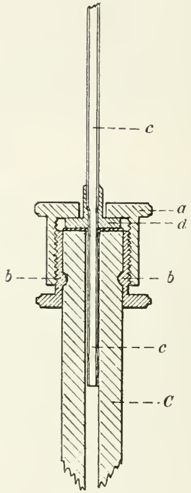

Fig. 2.

Joint between platinum capillary and neck of reservoir.

a, metal cap; *b*, split metal collar; C, porcelain capillary; *c*, platinum capillary, in this case drawn down to smaller size for lower 4 or 5 centims.; *d*, metal washer soldered to platinum capillary.

III. *The Scale, &c.*

The manometer scale, nearly two metres long, is graduated into millimetres and carries a slider with vernier and fine adjustment screw by which the position of the mercury column in B could be read directly to ·02 millim. Over the part of the scale used in these experiments no corrections were required.

Accurate setting of the mercury to the point in the closed limb of the manometer was facilitated by the use of a small short-focus telescope clamped to the stand. Measurements of the height of this point on the scale, as given by the vernier readings when the sighting edge of the slider was brought in line with the point, were taken frequently throughout the work. The transfer of level was effected by a Quincke microscope fixed on a plate-glass table which could be attached to the gas thermometer supports. This constant was different in the several sets owing to the tube A having for different reasons been several times rebuilt.

The Reservoirs and their Attachment, &c.

For some early experiments up to 200° C. reservoirs of normal Jena glass 16‴ were fixed having a long capillary stem attached by fusion. For the later series at high temperatures reservoirs of Berlin porcelain, similar to those formerly employed by CHAPPUIS and HARKER, were kindly ordered for us by Dr. HOLBORN at the Berlin factory. They were made in manner described by HOLBORN and DAY in their paper in 'Wied. Ann.,' vol. 68, pp. 830, 831.

For attachment of the reservoir to the platinum capillary leading to the manometer joints made by sealing-wax or cement are too treacherous to be trusted for long periods, so the joints were in all cases made in the manner shown in fig. 2. The brass washer soldered on to the platinum capillary is clamped down tight against the ground end of the porcelain or glass neck of the reservoir by a hollow cap which screws on to the two halves of a split brass collar, a thin leather washer, on which is smeared a little marine glue, being interposed to make the joint. The collar is prevented from sliding upon the neck by fitting closely into a rounded channel cut into the glass or porcelain by a small emery wheel. This form of joint has given perfect satisfaction.

being lost through the plug being forced outward by the mercury pressure. Although the possibility of the recurrence of this accident was avoided by the subsequent addition of a spring frame arrangement embracing the plug of the tap, it was found extremely difficult to arrange the lubrication to be free enough to permit of its constant use for shutting off communication between the thermometer and the bulb and at the same time to prevent very slow inward leakage of air into A through the tap when the bulb was under low pressure. The apparatus was therefore modified by putting into tube A a plain steel tap and introducing a smaller and perfectly ground three-way tap into the side tube, where it was not needed after the bulb was once filled. Under these conditions the whole arrangement could be kept perfectly gas tight.

IV. *Barometer and Auxiliary Measurements.*

The barometer used was a Fortin with $\frac{3}{4}$-inch tube, made by HICKS and standardised at Kew. It was compared from time to time with an old Royal Society's standard and found to have the same correction, −·05 millim.

The temperatures of the mercury columns in the gas thermometer were obtained by means of two auxiliary thermometers disposed at convenient heights, having their bulbs immersed in short test tubes filled with mercury. These are called (2) and (3) in the examples of gas thermometer readings given later. The temperature of that part of the dead space over the mercury in tube A is given by a thermometer inserted into the metal stopper carrying the reference point. This thermometer is labelled (1). The temperature of the platinum capillary is kept constant by surrounding it with a water-jacket of rubber tube, through which a circulation of water from a supply vessel kept in the room is maintained during the experiments, the temperature of the flow being given by the thermometer (4).

In making the correction of the pressures of the various mercury columns to latitude 45° and sea level, the latitude of the thermometric laboratory was taken as 51° 26′ 47″, and the height above sea level of the cistern of the barometer as 10 metres.

The relation

$$\frac{\text{G latitude Bushy House}}{\text{G latitude } 45^\circ} \text{ was taken as } 1{\cdot}000561.$$

V. *Preparation of the Gases.*

The nitrogen used in the earliest experiments was obtained from atmospheric air by the ordinary method of absorbing the carbonic acid by potash and the oxygen by hot copper, but for the sake of continuity with the work of CHAPPUIS and the author, in which chemical nitrogen was used in all the high-temperature experiments, the method of preparation employed by them was used in all the later work. Into a flask containing 100 grammes of potassium bichromate, dissolved in air-free distilled water, was introduced gradually by means of a tap-funnel a strong solution containing 100 grammes nitrate of soda and 100 grammes nitrate of ammonia. When gently heated on a water-bath this mixture gives off a steady stream of nitrogen, which is passed over hot copper and copper oxide to destroy any oxides of nitrogen it may contain, and collected over weak boiled potash solution in a gas holder. From this it passes through solutions of silver nitrate and strong potash, boiled sulphuric acid, and over solid caustic potash to a larger U-tube containing phosphoric anhydride, where it is left several days before use, or is collected in larger quantity in the dry gas holder over mercury. The tightness of all the joints and taps was ultimately made so perfect that no measurable inward leak of air could be detected by the attached manometer, even when the whole was left standing under reduced pressure for some months. A sketch of the gas preparation apparatus is shown in fig. 3.

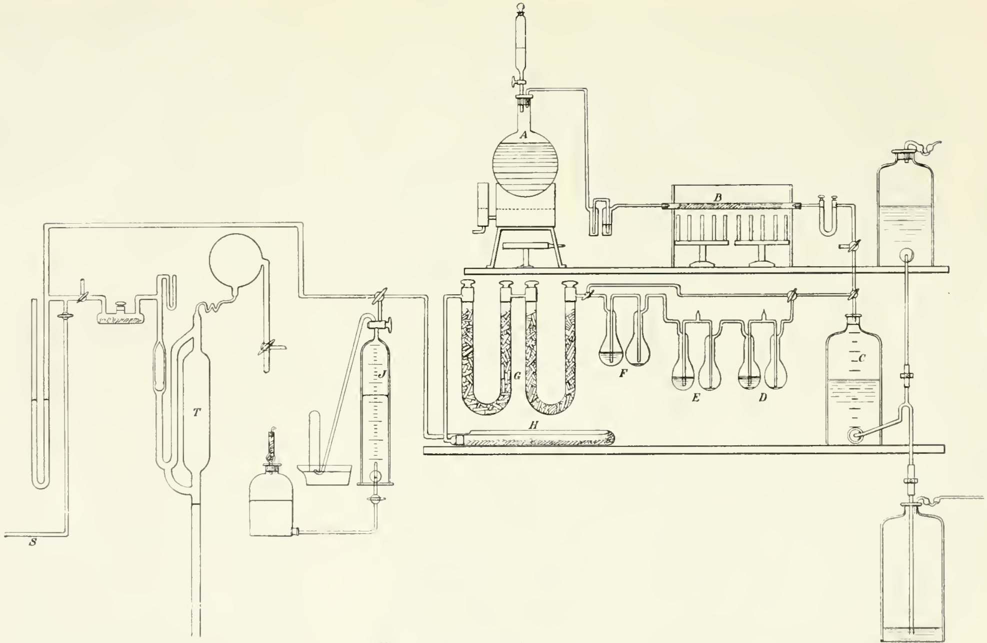

Fig. 3. Gas preparation apparatus.

A, generating flask ; B, tube containing copper or copper oxide ; C; wet gasholder ; D, silver nitrate solution ; E, strong caustic potash ; F, strong sulphuric acid ; G, solid potash and anhydrous baryta ; H, phosphoric anhydride ; J, mercury gasholder ; T, Töpler pump ; S, tube to gas thermometer.

VI. *Filling of the Reservoir.*

The experiments of Chappuis and the author, 'Phil. Trans.,' vol. 194, p. 93, showed how important it is that previous to and during the filling of the reservoir with gas the whole should be heated to as high a temperature as possible—indeed that with a reservoir of verre dur it was impossible to obtain an accurate value for a temperature in the neighbourhood of the boiling-point of sulphur unless it had been previously heated for a considerable period to a temperature considerably higher than this. Although very little is known as to the behaviour with change of temperature and pressure of the films of condensed gas, which by their surface tension adhere closely to the walls of a glass or porcelain vessel, and how they influence the coefficient of expansion of any gas enclosed in it, it seems at any rate advisable to remove the existing film in any reservoir to be used for high-temperature work as far as possible by repeated heating and evacuation. Accordingly for each new filling during this work the procedure was to heat the reservoir very strongly, exhausting it and the connecting tubes leading to the pump so thoroughly that an electrodeless vacuum-tube attached by a side tube showed strong green fluorescence both when the reservoir was hot and the next day after cooling. This was only attained after much labour by taking special care in the manipulation of all the taps and joints in the circuit, and by blowing together into one piece the whole of the long canal leading from the gas thermometer in the main laboratory to the pump and gas preparation apparatus in a small room on an upper floor. After the apparatus had stood this test the gas was filled into the reservoir to 100 to 200 millims. pressure at least three times in each case before the final filling and pressure adjustment were made, the bulb being heated meanwhile as strongly as expedient in the particular case.

VII. *Dilatation of the Porcelain Reservoir.*

In 1898 and 1899, Chappuis and Harker, in the absence of precise data as to the expansion at high temperatures of the reservoirs of Berlin porcelain they employed in their gas thermometer, were obliged to content themselves with extrapolation of the expression for the expansion obtained by them from determinations made in the Benoit-Fizeau dilatometer between the limits 0° and 100°, confirmed by further experiments made by a weight thermometer method. The values for the expression they employed seemed at the time to be amply confirmed by the measurements of Holborn and Wien, made in 1892, which gave for the mean coefficient of linear expansion between the limits 0° and 600° the value 4400×10^{-9}, the value calculated by Chappuis and Harker for the same temperature limits being 4484×10^{-9}. On the other hand, the later experiments of Holborn and Day made on Berlin porcelain rods heated horizontally in an electric furnace show fairly conclusively that the expansion of porcelain cannot be represented by any simple function for more than a few hundred degrees, and that therefore an expression deduced empirically cannot be applied outside the limits of actual experiment without risk of serious error.

To obtain the most probable value over the whole range for the expansion of Berlin porcelain, a combination was made of the values obtained directly by two independent methods over the range 0° to 100° C. by CHAPPUIS and HARKER with those obtained for higher temperatures by HOLBORN and DAY. As pointed out by CHAPPUIS in the paper in the 'Philosophical Magazine' alluded to in the introduction, the formula of HOLBORN and DAY undoubtedly gives too high values below 250° C. This has just been confirmed by the experiments of SCHEEL, published in vol. 4 of the 'Wiss. Abh. der Reichsanstalt,' whose results over the range 14° to 100° obtained in the Fizeau dilatometer are in close agreement with those of CHAPPUIS and HARKER.

I have therefore taken for this work the following values of the mean dilatation between 0° and T°, and from them calculated a table giving the volume at T° of a reservoir, whose volume at 0° is unity, for temperatures up to 1100° C.

MEAN Dilatation of Berlin Porcelain between 0° and T°.

T.	Dilatation. $\times 10^{-8}$.	T.	Dilatation. $\times 10^{-8}$.
°			
0	269	600	363
100	299	700	374
200	318	800	385
300	329	900	397
400	340	1000	408
500	352	1100	419

whence volume at T° of reservoir whose volume at 0° = 1

0	1·000000
100	1·000897
. . .	. . . . .
. . .	. . . . .
1000	1·013834, &c.

VIII. *Pressure Coefficient of the Reservoirs.*

In the work of CHAPPUIS and HARKER account was taken of the change of volume of the bulb produced by changes of internal pressure. The value of the change of volume was deduced from observations made by varying the external pressure on the reservoir at ordinary temperatures. With the thick-walled reservoirs here employed the whole effect is only very small, a change of 1 metre in the internal pressure producing a change of about $\frac{1}{40,000}$ of the whole volume of the reservoir at ordinary temperatures. The effect of high temperatures in changing the elastic constants of the porcelain here involved being quite unknown, it was deemed justifiable to omit this correction altogether for the present high-temperature experiments.

IX. *The Platinum Thermometers.*

The platinum thermometers chosen for the comparison at high temperatures with the gas thermometer were one of a group of six constructed in 1902 at the laboratory for the British Association and described as BA_2 in the report of the Electrical Standards Committee for 1903, and the much older one, lettered K_9, which had been employed by CHAPPUIS and HARKER in their work at Sèvres. Since the date of the Sèvres comparisons in 1898 and 1899 this thermometer had been preserved as a standard and used comparatively seldom.

Its characteristic dimensions are :—

Length from end of porcelain tube to below the wooden collar, 35·5 centims.
External diameter of tube, 11·5 millims.

The head was of the old pattern with terminals, the leads being entirely of platinum, ·020 inch diameter, and the "bulb" wire, ·006 inch diameter.

Thermometer BA_2 is similar in design to CHAPPUIS' and HARKER'S thermometers K_8 and K_9, illustrated in their paper, the head being arranged to be capable of standing a vacuum for a considerable time.

Its dimensions were as follows :—

Length of tube to under head = 43·5 centims.
Internal diameter of porcelain tube = 11·5 millims.
Length of mica cross = 40 millims.
Thickness of "bulb" wire = ·006 inch.
,, "lead" ,, = ·020 ,,

Both "bulb" and leads were constructed from the stock of wire of extra high purity prepared by JOHNSON and MATTHEY for the British Association. Details as to the construction of these British Association thermometers are given in the appendix to the Report of the British Association, 1903.

X. *The Resistance Box for Platinum Thermometry.*

The resistance box used was the one originally described by GRIFFITHS in 'Nature,' November 14, 1895, pp. 39 to 46, which had subsequently been modified by the Cambridge Scientific Instrument Company by the substitution of Doulton-ware plug holders for the usual brass blocks to which the individual resistances were connected. In this form the box was used by CHREE in his investigations described in 'Proc. Roy. Soc.,' vol. 67, pp. 3–58.

During the period covered by the present work the coils were standardised in the manner described by HARKER and CHAPPUIS (*loc. cit.*, p. 52), and great care was taken throughout as to the cleanliness of the plugs and to prevent the accumulation of the black deposit which has a tendency to form in the plug holes. The box unit is very exactly ·01 ohm, and the nine coil values are arranged on the binary scale from 5 up to 1280 units, with an additional coil of 100 units for the determination of the fundamental interval. The coils are of fine platinum-silver wire wound into loose hanks not paraffined, so as to follow with as little lag as possible the temperature of the massive copper tank in which they are placed. The working of the resistance box has under the conditions which obtained for this work been sufficiently good to render unlikely the introduction of any systematic errors. The sensitivity of the platinum thermometer bridge was such that even at the highest ranges a *change* of about ·01° C. could be detected, though an accuracy of ·1° at 1000° C. is probably about the limit in the *absolute* measurement of temperature even under favourable conditions with resistance coils of platinum-silver, such as those employed here, with a temperature coefficient of ·00026 per degree C.

The flexible leads which were never detached from the thermometers during any series were carefully checked before and after the experiments, and the possible checks, got by using different combinations of coils in measurement of the same temperature, always agreed reasonably well.

XI. *The Potentiometer for Thermocouple Measurements.*

The potentiometer used in these experiments was specially designed and made in the laboratory for accurate thermocouple work, and was described in detail in a paper in the 'Phil. Mag.' for July, 1903. It is direct reading, and is arranged so that for ordinary thermoelectric work where the maximum electromotive force to be measured is of the order of ·02 volt the individual settings could be made to about ·1 microvolt if required. A diagrammatic representation of the connexions is shown in fig. 4 and

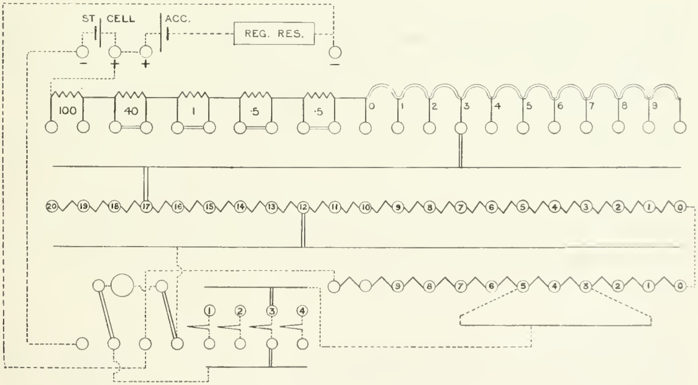

Fig. 4. Connections to potentiometer.

a plan of the top of the instrument in fig. 5. It is designed with a view of rendering possible the use of a short slide-wire of large cross section. The balancing coils, on which the fall of potential is adjusted to a definite value, are in two rows, the centre row of the box consisting of 20 coils of $\frac{1}{10}$ of an ohm each. In series with these is a second row, immediately behind the bridge-wire, consisting of 11 coils of $\frac{1}{100}$ of an ohm each. By means of an arrangement of thick copper bars connected with the ends of the slide-wire, which has a total resistance of ·02 ohm, any two adjacent coils of this latter series may be put in parallel with the slide-wire. The 11 coils of ·01, two of which are thus shunted, are therefore always exactly equivalent to ·1 ohm.

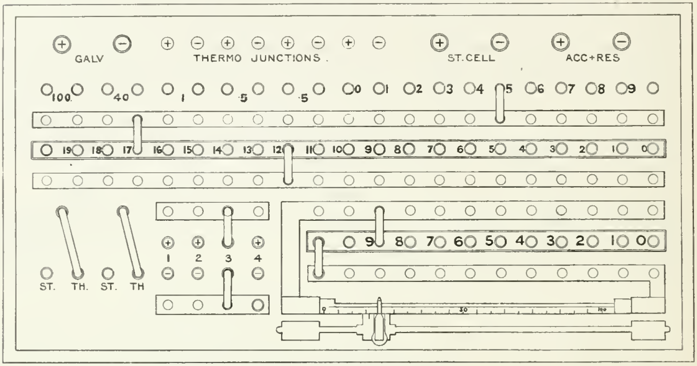

Fig. 5. Plan of potentiometer.

For ordinary thermoelectric work the fall of potential along these two sets of coils is adjusted so that each of the back row represents 1000 microvolts, each coil of ·01 ohm being therefore 100 microvolts. The slide-wire, 200 millims. long, is provided with a divided scale on which $\frac{1}{1000}$ part of its length can be easily estimated. It will be seen that the slide-wire thus connected acts like a vernier to the small coils.

The adjustment of the electromotive force is made by a standard Clark or Weston cell and the auxiliary set of coils in the back row, a feature of the instrument being that without any external alteration either form of standard may be used at will. The five coils to the left are permanently connected in series, but are arranged so that any coil may be cut out of circuit when required. Their values are 100, 40, 1, ·5, and ·5 ohms respectively. Those to the right are a set of 10 simple series coils of ·01 ohm each, arranged so that a connexion can be taken from any one of them to the long copper bar just in front. Suppose we wish to use as a standard a Clark cell whose electromotive force at the prevailing temperature is 1·4333 volts. It is obvious that we may make the fall of potential over the ·1 ohm balancing coils have the desired value of 1000 microvolts by putting into circuit coils 100, 40, 1, ·5, and three of the ·01 series, leading from the third hundredth by means of the copper bar to No. 17 of the balancing set, when altogether we shall have

$$\left.\begin{array}{r}100\\40\\1\\\cdot5\\\cdot03\end{array}\right\}\text{ in the back row,}$$

$$\left.\begin{array}{r}1\cdot7\\\cdot1\end{array}\right\}\text{ in balancing coils and bridge-wire,}$$

making in all 143·33 ohms.

Should a Weston cell having an electromotive force of 1·0186 volt be substituted for the Clark, the only alteration necessary would be to short-circuit coils 40, 1, and ·5, and to move the connector from the third to the sixth of the set of hundredths. The compensating current is furnished by a small secondary cell, in series with which is a dial resistance capable of variation up to 200 ohms by steps of ·005 ohm.

The four thermojunction circuits provided are connected to a selector switch, by means of which each successively or any two of them connected in opposition may be brought into circuit, and the change-over from the standard cell connexion required in the preliminary adjustment is made at the two-way switch at the front left-hand corner, by means of which the galvanometer may be put into the circuit desired.

Constructional details are given in the paper referred to. All the coils employed are of selected manganin, carefully annealed, and all connexions are made by mercury caps and copper short-circuiting pieces, the only metals employed anywhere in the parts carrying current being copper and manganin.

The values of those coils in the box which were used in this work as determined at the conclusion of the comparison experiments are given in the following table. Any alteration in their relative values which had taken place since the first standardization is undoubtedly so small as to be quite negligible compared with other errors in thermocouple work.*

---

* It is obvious that in building down to obtain a convenient standard of thermal electromotive force—100 or 1000 microvolts—so long as the *relative* values of the coils employed remain the same, their absolute value is of no moment.

A matter of great importance, however, is to measure the value of each coil in exactly the same way as it is used. Accordingly, for this standardization a potential method of measurement was employed in all cases, the current and potential leads being connected exactly as in actual work.

Coil Values in ohms at 17° C.

Coils in main row nominal value ·1 each.

No. 1	·100011	No. 6	·100022	No. 11	·100019	No. 16	·100026
„ 2	·100028	„ 7	·100046	„ 12	·100026	„ 17	·100025
„ 3	·100025	„ 8	·100026	„ 13	·100023	„ 18	·100022
„ 4	·100021	„ 9	·100019	„ 14	·100025	„ 19	·100029
„ 5	·100022	„ 10	·100022	„ 15	·100020	„ 20	·100033

Mean value of the 20 tenth-ohm coils = ·100024.

Total „ „ slide-wire set of 11 coils shunted by slide-wire = ·100043.

Maximum variation in the resistance of slide-wire set for different positions of the slide-wire connector = ·000008.

Value of nominal 100-ohm coil = 100·029.

The auxiliary set of ·01 ohm coils for temperature compensation were all found to be within $\frac{1}{1000}$ part of their face value.

XII. *Standard of Electromotive Force.*

The standards of E.M.F. used with this potentiometer were two similar **H**-form cadmium-sulphate cells with saturated solution of the type employed at the Reichsanstalt, and made up as part of a large batch of similar ones by Mr. F. E. Smith in 1902. From Mr. Smith's measurements as to the relation of the E.M.F. of these cells to the standard Clark cells of the laboratory, and from other data, it is practically certain that the error committed in assuming their E.M.F. to be identical with those at the Reichsanstalt is not greater than 1 part in 10,000, which corresponds to a tenth of a degree at 1000° C. with the thermojunctions employed. For the E.M.F. of each of these cells, which throughout the work were never found to differ by more than ·0001 volt, the Reichsanstalt official value, namely 1·0186 volt at 20° C., was assumed.

The following table gives the value of the total resistance used in the potentiometer at different temperatures to adjust the E.M.F. over each coil of the main row to exactly 1000 microvolts :—

Temperature . .	5°.	10°.	15°.	20°.	25°.
Resistance. . .	101·90	101·89	101·88	101·86	101·84

The table of the values of the coils in the potentiometer shows that the actual resistance corresponding to a nominal value of 101·86 ohm is 101·889. The relation of this to the mean value of the main set of tenths, namely ·100024 ohm, is well within 1 part in 10,000 of its nominal value, and an inspection of the table shows that the cumulative effect of individual coil errors is very small. No corrections were therefore applied to any of the potentiometer readings.

Special precautions were taken to avoid the effect of temperature variation both on the accumulator furnishing the compensating current of ·01 ampère and on the standard cells, both being placed in double-walled boxes surrounded by a thick layer of cork clippings. Under these conditions the daily temperature range in the boxes was found to be reduced to about one-fifth of the value outside them, and the compensating current could generally be kept to within $\frac{1}{10,000}$ of its value for an hour at a time without adjustment.

XIII. *The Thermojunctions.*

The thermojunctions used in this research were composed of pure platinum with platinum containing 10 per cent. of rhodium, and were all ·6 millim. diameter. They were obtained from HERÄUS of Hanau, through Dr. HOLBORN of the Reichsanstalt, and were compared by him with the standard junctions of the Reichsanstalt at a number of fixed points.

In a letter to the Director of the Laboratory, Dr. HOLBORN says: "Two elements were compared at four points, the melting-points of Zn, Sb, Ag, and Cu, and gave the following results[*] :—

	Temperature.	Microvolts.
	° C.	
Zinc	419·0	340_9
Antimony	630·5	550_4
Silver (in graphite). . .	961·5	908_3
Copper (in air)	1065·0	1027_7

Before commencing comparisons with the gas thermometer, three independent determinations of the freezing-point of silver were taken in an electrically heated crucible furnace with one of these junctions N.P.L. 2; in these experiments two different observers took part, and the three results were :—

$$\left.\begin{array}{r}9087\\9092\\9082\end{array}\right\}\text{microvolts.}$$

The mean of these, 9087, agrees very closely with the datum given above. Junction N.P.L. 2 was selected for the comparison, while N.P.L. 1 was reserved as a master standard, and was compared with No. 2 in a specially arranged electric furnace at temperatures up to 1200° C. before and after the investigation.

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[*] In all experiments with thermojunctions here referred to, it is to be understood that the cold junction is at 0° C.

The results of these comparisons show conclusively that Junction No. 2 has not suffered any material alteration during its protracted heatings at high temperatures.*

The E.M.F. of the two junctions, as given by a comparison made at the close of the work, is shown in the following table :—

Approximate temperature.	Microvolts.		Difference, 1 – 2.	Difference in ° C.
	No. 1.	No. 2.		
290°	2202·3	2202·1	+ ·2	+ ·02
385	3089·1	3089·4	– ·3	– ·03
473	3930·6	3930·3	– ·3	– ·03
489	4088·1	4084·8	+3·3	+ ·34
628	5474·3	5471·3	+3·0	+ ·30
795	7254·5	7249·3	+5·2	+ ·48
859	7937·5	7931·5	+6·0	+ ·54
1152	11308·5	11303·8	+4·7	+ ·39

XIV. *Formulæ for Thermojunctions.*

From the values of the E.M.F. of N.P.L. 1 and 2, as determined above by HOLBORN, a formula involving two powers of the temperature was calculated by least squares to give the relation between E.M.F. and temperature to represent the Reichsanstalt's scale.

The formula

$$E_t = -304 + 8{\cdot}165\,t + 0{\cdot}001663\,t^2$$

gives residuals at the four melting-points given above much smaller than their probable error.

The corresponding formula for HOLBORN and DAY's own standard junction T_2, using only the values of the temperature obtained from the gas thermometer with bulb of platinum-iridium and employing the revised data for the expansion of this material at high temperatures,† is

$$E_t = -310 + 8{\cdot}048\,t + 0{\cdot}00172\,t^2.$$

The following table gives side by side the E.M.F. of HOLBORN and DAY's junction T_2, and of our own junctions at temperatures above 300° :—

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* At the conclusion of the second set of comparisons the metallic lustre of some inches of the platinum wire was decidedly impaired, due probably to the natural disintegration of the material, but this did not appear to be accompanied by the smallest change in the E.M.F. of the junction.

† HOLBORN and DAY, 'WIED. Annalen,' 1900, vol. 2, p. 520. The change from the original formula for the expansion of the bulb involves a correction of the scale amounting to 4° at 1000°.

T.	NPL 1 and 2.		HOLBORN and DAY'S T_2.		Difference NPL − T_2.
300	2295		2260		35
		933		925	
400	3228		3185		43
		966		960	
500	4194		4145		49
		1000		994	
600	5194		5139		55
		1032		1029	
700	6226		6168		58
		1066		1063	
800	7292		7231		61
		1099		1097	
900	8391		8328		63
		1133		1132	
1000	9524		9460		64
		1166		1166	
1100	10690		10626		64

It will be observed that both the formulæ just quoted do not apply at lower temperatures, being nearly 40° C. in error at 0°, and that therefore extrapolation downwards even over a narrow range is not permissible. The error of the formula for NPL 2 was determined to be 4° at 200°.

XV. *Determination of the Fixed Points* 0° *and* 100° *and Sulphur Boiling-point.*

The determinations of the fixed points 0° and 100° for the gas and platinum thermometers were made in baths specially built for each kind of instrument. The ice-points were taken in glass vessels of a capacity of about 6 and 3 litres respectively, consisting each of an inverted glass bell-jar with draining arrangement below, and surrounded by a thick packing of cork clippings. Very little melting of ice took place even in 12 hours, the upper surface of the ice being protected by a thick felt covering wrapped round the stem of the instrument.

The block-ice previous to use was always well washed and finely divided by an ice plane, and was repeatedly tested for dissolved impurity, the method adopted being to ascertain the amount of chlorine present by addition of silver nitrate to the drainings. It was found satisfactory, except on one occasion.

The steam-point apparatus for the gas thermometer was of the usual type with concentric tubes, but was arranged to be easily changed from the vertical to the horizontal position by suitable couplings of wide compo tubing, connecting it to the boiler. The steam- and zero-points for any series of comparisons were always taken with the reservoir in the position in which it was used in that series, and in all cases the amount of stem emergent was made as nearly as possible the same as in the comparisons. The steam bath as arranged for the horizontal position is shown in fig. 6. The bulb of the gas thermometer, resting on a small piece of cork, occupies the centre of the inner tube, through which steam brought direct from the boiler by a well protected wide tube is circulated. The steam issuing from the outer jacket is condensed and returned to the boiler as shown. The excess of the steam pressure over that of the atmosphere, which in these experiments was seldom over 1 millim. of water, is indicated by a small graduated water gauge.

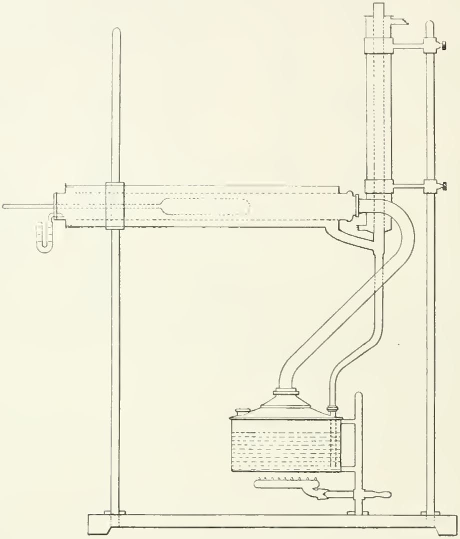

Fig. 6. Steam bath.

The sulphur-points of the platinum thermometers were taken in the usual manner in the well-known Callendar form of boiling-point apparatus. The only departure from previous custom being that for the glass boiling tube was substituted one of thin weldless steel,* which is more durable and can be heated up quickly without being removed from its asbestos cover. Careful comparisons of this form with the older glass apparatus showed no systematic discrepancy.

For the sulphur boiling-point CALLENDAR's old value 444·53° C. at normal pressure was taken. The sulphur-points were always taken on days when the pressure was not far removed from 760 millims. to eliminate the uncertainty as to the coefficient to use when reducing to normal pressure.

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* An ordinary iron tube such as gas or steam pipe cannot be used for this purpose, as owing to conduction from the flame of the burner upwards there is a tendency for the sulphur vapour to become superheated.

XVI. *Electric Furnaces.*

Two different electric furnaces were employed in this work. Their dimensions were similar, but they differed in that in the first the heating-wire was wound uniformly, and in the second an approximation to a logarithmic spiral was made at each end, the turns being gradually crowded, so that the cooling effect of the ends was in a great measure compensated by the additional heat supply. Both furnaces were wound with wire of pure nickel about 1·6 millims. diameter. The heating current was supplied from a special battery of 56 accumulators reserved for this purpose, which was divided into four groups of 14 cells, capable of being coupled in series or parallel, as desired. A set of large well-ventilated manganin resistances, formed of two No. 9 wires in parallel, and capable of carrying 100 ampères without undue heating, was arranged so that the external resistance of the circuit could be altered by steps of ·025 ohm up to 3·2 ohms, thus enabling any desired amount of energy to be put into the furnace at will. The construction of the furnace and the disposition of the different instruments within it is shown in fig. 7. The nickel heating-wire is wound upon the inner tube of unglazed biscuit porcelain, and in order to prevent the turns becoming short-circuited when hot, the whole of the wound portion is covered with a thin layer of "purimachos" which is baked on at a moderate heat. The leading-in wires are doubled or trebled in all cases. The bulb of the gas thermometer is supported on a small bridge of fire-clay resting on the furnace bottom, and the standard platinum thermometer and thermojunction are arranged as shown, great care being taken that neither the wires of the junction nor the thin porcelain

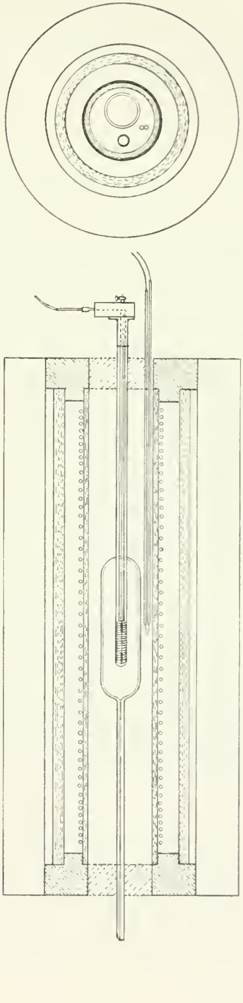

Fig. 7. Electric furnace, longitudinal and transverse sections.

capillary tubes used to cover them shall anywhere touch the furnace wall. As previous experience with gas and platinum thermometers, whose walls were of porcelain, had shown how very much more slowly the transfer of heat took place through this material than through metal or glass, even when surrounded by a stirred liquid, it was judged preferable to make a comparatively small number of high-temperature experiments, in which great constancy of temperature was attained for some time previous to and during the observations, rather than to attempt to obtain mean values from more extended series under less perfect conditions. With this object, a Callendar recorder was connected to a second platinum thermometer placed in the furnace, and, during the adjustment of the temperature and the comparisons, records from this instrument were taken on an open scale. The use of the recorder greatly shortened the time necessary for the establishment of a steady temperature by guiding the observer as to the manipulation of the resistances in the heating circuit. In addition to the large set of resistances in the heating circuit, a set of coils of ·01 ohm each was placed close to the recorder, and it was found that a change of one step on this set made all the difference between a steady state and a gradual rise or fall in the temperature of the furnace when equilibrium had been nearly established. When desired, it was quite easy to keep the furnace temperature constant to about a fifth of a degree for half-an-hour at a time, at temperatures as high as 1000° C., but in most of the experiments the temperature was intentionally allowed to rise very slowly. Without these precautions, comparisons between instruments of such widely differing "lag" as a bare thermojunction and a gas thermometer with porcelain reservoir, whose walls were 2 millims. in thickness, would have undoubtedly been liable to serious error.

XVII. *Furnace Correction.*

In order to investigate the distribution of temperature throughout the space filled by the gas thermometer bulb, which was about 130 millims. in length, a pair of thermojunctions, quite independent of the standard, were arranged so as to measure the temperature difference between the centre of the furnace and points further out, and so obtain a correction to be applied to the readings of the gas thermometer, to reduce its indications to what it would have registered had the whole of the bulb been at the same temperature as the middle point.

XVIII. *Exploration of Furnace.*

For this purpose a thin wire junction of platinum with platinum iridium was chosen on account of its great sensitiveness at high temperatures. This was made up to work differentially, and was composed of a piece of platinum iridium between two pieces of platinum, thus forming two junctions, which were both placed in the hot space, entering the furnace from opposite ends. The wires were stiffened by threading them through thin porcelain capillary tubes, and the two junctions of the platinum with the copper forming the rest of the circuit were placed together in ice.

The E.M.F. given by this element for a difference of 1° between its two hot junctions is given in the table below.* This is obtained by direct comparison over the whole range of a simple junction made up of wires from the same reels, which had been similarly treated.

Temperature in °C.	Difference for 1° in microvolts.
500	16·4
600	16·6
700	16·8
800	16·9
900	17·1
1000	17·2

One of the junctions was carefully placed and kept at the middle point of the gas thermometer bulb, and the second was arranged so that it could be pushed backward and forward into positions 2, 4, 6, and 8 centims. to the right and left of this point, observations being made of the differential E.M.F. produced in each of these positions.

From a number of such observations made after the different comparisons and spaced over the interval 400° C. to 1000° C., curves were constructed showing the distribution of temperature over this space for each of the furnaces used, and by measuring these curves the difference between the average temperature of the bulb and that of the central point, where the standard thermojunction was placed, could easily be found.

For the compensated furnace the mechanical centre was found not to quite coincide with the position of highest temperature at the higher ranges, though over the lower part of the scale the curves were practically symmetrical.

The corrections obtained by this method from the different explorations were plotted as ordinates against furnace temperatures as abscissæ, and from the mean curve thus obtained the following values were deduced as the mean corrections at different temperatures:—

---

* In practice, when the junctions were placed as close together in the furnace as possible without actual contact, owing to the inevitable small secondary effects arising from unsymmetric heating of the junction wires, the E.M.F. round the circuit was not always zero. When, however, the junctions were so supported as to nowhere touch the furnace wall, the total effect was always quite small and was allowed for in each case.

Furnace temperature.	Corrections to gas thermometer reading for compensated furnace experiments.
° C.	° C.
500	+1·7
600	+1·5
700	+1·3
800	+1·1
900	+ ·8
1000	+ ·0

From this table the numbers in the column headed "furnace correction" in Series II. and III. have been deduced by interpolation. For the uncompensated furnace used in the earlier experiments the correction is uncertain to about 0·5° C., and the results are, therefore, only given to this accuracy in the final column representing the "corrected gas thermometer temperature."

XIX. *Method of Calculation.*

The observations taken with the gas thermometer were calculated according to the usual formula.

Let H_0 = pressure at 0°,
H = ,, ,, T°,

and let t_1 and t'_1 be temperatures of "dead space" with bulb at 0° and at temperature T°.

Let $\frac{v}{V_0}$ = relation of the volume of the whole "dead space" to the volume of the bulb at 0°,

α = coefficient of expansion of the gas at constant volume between 0° and 100°,

3β = the mean cubical coefficient of expansion of the porcelain bulb between 0° and T°;

then

$$H_0\left(1+\frac{v}{V_0}\frac{1}{1+\alpha t_1}\right)=\frac{H}{1+\alpha T}\left\{1+3\beta T+\frac{v}{V_0}\left(\frac{1+\alpha T}{1+\alpha t'_1}\right)\right\},$$

therefore

$$T=\frac{H}{\alpha H_0\left(1+\frac{v}{V}\frac{1}{1+\alpha t_1}\right)}\left\{1+3\beta T+\frac{v}{V_0}\left(\frac{1+\alpha T}{1+\alpha t'_1}\right)\right\}-\frac{1}{\alpha}.$$

For a given filling $H_0\left(1+\frac{v}{V_0}\frac{1}{1+\alpha t_1}\right)$ is a constant if the zero-point remains constant.

The values of α obtained from the steam-point determinations were calculated from the same formula, inserting the value of T obtained from the Regnault-Broch table for the boiling-point of water under the prevailing pressure, as given in the example.

XX. *Series II.*

ICE-POINT; 3. April 16, 1903.

Observer, J. A. H.

Time.	Scale reading.	Dead space.* (1.)	Mercury columns. (2.)	Mercury columns. (3.)	Barometer.	Barometer temperature.
h. m.						
2 20	159·86	13·8	14·2	14·0	768·73	15·0
23	159·98	13·8	14·2	14·0	·62	15·0
26	160·00	13·9	14·3	14·05	·58	15·2
30	160·00	14·0	14·3	14·15	·60	15·2
	159·96	13·9	14·25	14·05	768·63	15·1
Index corr. =	747·18	− ·0	− ·00	− ·00	Lat. corr. = + ·43	− ·1
					Temp. corr. = − 1·88	
	587·22	13·9	14·25	14·05	Scale corr. = − ·05	15·0
Temp. corr. =	− 1·35					
Lat. corr. =	+ 0·32				Corr. bar. = 767·13	
		Merc. col. temp.	14·15°			
	586·19					
	767·13					
H$_0$ =	180·94					

It will be observed that during the 10 minutes covered by the observations the barometric pressure fell rather more than 0·1 millim. while the scale reading rose a corresponding amount.

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* On this occasion the variations in the temperature of the jacket water were so abnormally great during the preliminary period before the experiment, that it was decided to dispense with the circulation in the space round the platinum capillary and take the whole dead space as being at the temperature of the part below the stopper in the closed limb of the manometer, given by thermometer No. 1.

STEAM-POINT; 3. April 16, 1903.

Observer, J. A. H.

Time.	Scale reading.	Dead space. (1.)	Mercury columns. (2.)	Mercury columns. (3.)	Barometer.	Barometer temperature.
h. m.						
3 15	225·58	14·2	14·55	14·30	768·60	15·5
22	·63	14·2	14·65	14·35	·62	15·5
30	·67	14·35	14·70	14·50	·60	15·5
32	·58	14·45	14·70	14·50	·55	15·5
	225·61	14·3	14·65	14·4	768·59	15·5
	747·18	– ·0	– ·0	– ·0	Lat. corr. = + ·43	– ·1
					Temp. corr. = – 1·94	
	521·57	14·3	14·65	14·4	Scale corr. = – ·05	15·4
Temp. corr. =	– 1·23					
Lat. corr. =	+ 29				Corr. bar. = 767·03	
		Col. temp. taken as	14·5°			
	520·63					
	767·03					
					millim.	millims.
		Corr. bar. . . . . =				767·03
H =	246·40	Water manometer on steam bath =			+ 1 =	+ ·07
		Total pressure on steam . . .				767·10

Boiling-point of water at 767·10 millims. = 100·26°.

$\alpha = \cdot 003669_1$.

XXI. *Accuracy of Gas Thermometer Constant Determinations.*

As an example of the kind of accuracy attained in the determination of the constants of the gas thermometer, the individual values of the zero- and steam-points taken before the comparisons in Series II. is given along with the readings taken in one complete determination of each.

For calculation of experiments of this kind, where no gradual systematic drift is expected, instead of utilising for calculating the individual values of the α the single determinations of the ice-point, the mean of the ice-points is taken for this purpose.

The difference between the value of α found before and after the comparisons is within the limits of experimental error. It will be seen from the table appended, which gives in an abridged form the calculation of the gas thermometer temperatures in the 13 experiments of Series II., that the difference has, however, been treated as real, and assumed to vary with the time. A more serious change has, however, taken place in the quantity of gas present in the bulb. The change in this case has also been assumed as proportional to the time, since before the first experiment was made the whole furnace was maintained at a high temperature for some time, and there is no sufficient evidence that the change occurred otherwise than regularly. So far as could be found by testing on the pump after the conclusion of the series no leak could be discovered, even when the bulb was at a full red heat.

CONSTANTS for Gas Thermometer. Series II.

I. Before the comparisons :—

Ice-points.

	H_0.	Dead space temperature.
(1)	180·90	12·6
(2)	·93	13·1
(3)	·94	13·9
Mean ice-point before comparison. . =	180·923	

Steam-points.

	H_{100}.	Dead space temperature.	α.
(1)	246·24	13·95	$\cdot 003669_9$
(2)	·26	13·75	$\cdot 003671_1$
(3)	·23	14·3	$\cdot 003669_1$
		Mean value of α before comparisons =	$\cdot 003670_0$

II. After comparisons :—

Ice-points.

	H_0.	Dead space temperature.
(4)	182·67	17·4
(5)	182·65	17·8
Mean ice-point after comparisons. . =	182·66	

Steam-points.

	H_{100}.	Dead space temperature.	α.
(4)	248·64	16·3	$\cdot 003671_5$
(5)	248·63	16·6	$\cdot 003670_8$
			$\cdot 003671_2$

XXII. *Summary of Calculation of Gas Thermometer Temperatures.*

Series II.

No. of experiment.	Scale reading.	Index correction.	Differential pressure.	Mean temperature of manometer column.	Corrected differential pressure.	Barometer reading.	Barometer temperature.	Corrected barometer reading.	Mean dead-space temperature.	H.	$H_0\left[1+\frac{v}{V}\left(\frac{1}{1+\alpha t_1}\right)\right]$.	α ·00.	Gas thermometer reading.
1	552·43	747·18	−194·75	15·8	−194·36	758·62	15·7	757·05	14·4	562·67	182·636	3670	597·9
2	664·01	747·18	− 83·17	16·3	− 83·00	757·23	16·2	755·60	15·4	672·60	182·781	3670	775·6
3	581·92	747·18	−165·26	12·8	−165·01	755·55	11·9	754·45	12·7	589·44	182·926	3670	639·7
4	612·32	747·18	−134·86	13·5	−134·63	751·36	12·8	750·15	13·6	615·52	183·070	3670	681·1
5	515·34	747·18	−231·84	15·1	−231·41	749·70	15·3	748·20	15·0	516·79	183·216	3670_5	521·4
6	734·00	747·18	− 13·18	16·6	− 13·16	749·72	16·45	748·08	15·6	734·92	183·361	3670_5	874·2
7	786·11	747·18	+ 38·93	17·3	+ 38·84	750·00	17·0	748·28	16·5	787·12	183·505	3670_5	959·5
8	562·43	747·18	−184·75	16·4	−184·36	751·40	16·55	749·75	16·3	565·39	183·650	3670_5	597·0
9	699·39	747·18	− 47·79	15·1	− 47·71	752·08	14·4	750·68	15·0	702·97	183·795	3670_5	819·0
10	540·90	747·18	−206·28	15·9	−205·87	754·96	15·95	753·37	15·7	547·50	183·940	3671	566·9
11	711·16	747·18	− 36·02	15·5	− 35·95	748·52	14·3	747·14	15·4	711·19	184·085	3671	830·4
12	733·52	747·18	− 13·66	14·6	− 13·64	748·17	13·8	746·86	14·4	733·22	184·230	3671	865·8
13	816·95	747·18	+ 69·77	14·9	+ 69·64	749·92	13·9	748·59	14·8	818·23	184·375	3671	1005·0

The "index correction" was determined on two occasions during this series, the individual readings being :—

747·16
747·19
747·18
747·19
747·17

Mean = 747·18

The relation $\frac{v}{V} = \cdot010_0$.

XXIII. *Method of calculating Comparison Experiment.*

As an example of the method employed in making and calculating out an experiment, the results of the observations made in Experiment No. 13 of Series II. are given in full. During this experiment the steadiness of the temperature of the furnace was perhaps a little above the average, but the room temperature and that of the resistance-boxes and of the various mercury columns of the gas thermometer and barometer were rising more rapidly than usual. According to observations taken on the thermojunctions and platinum thermometer, the steady state had been maintained for about 10 minutes before the first recorded reading on the gas thermometer was taken. As only two observers were available, one of these took alternate observations on the gas thermometer and thermojunctions, while the second took the platinum thermometer readings.

By graphic interpolation the mean thermojunction readings, corresponding to the times at which the other instruments were read, were obtained and are given in the example.*

The readings of the gas thermometer are made as independent of one another as possible by lowering the mercury each time before a setting, raising it again slowly so as to make a new meniscus.

The calculation to the accuracy here necessary of the air temperature corresponding to definite platinum temperatures is somewhat laborious, if the formula

$$\mathrm{T} = \left(\frac{5000}{\delta} + 50\right) - \sqrt{\left(\frac{5000}{\delta} + 50\right)^2 - \frac{10{,}000\,pt}{\delta}}$$

has to be applied for each observation. The graphic methods used by HEYCOCK and NEVILLE cannot easily be made sufficiently accurate. Since the value of δ for pure platinum wire has been found to be 1·5, varying from one specimen to another within very narrow limits, the most suitable method of effecting this conversion was found to be to construct a table giving, for a sufficient number of points, the value of T for given values of pt when $\delta = 1{\cdot}5$. A second table gives the correction to apply to the value of T thus obtained, if the δ differs from the standard value by a small amount.

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* The letters AB, BA refer to the position of the reversing switches leading to the potentiometer. Owing to small Thomson and Peltier effects, there was generally a small difference between the two positions. In this case it is rather above the average. There was no difficulty in setting to ·1 microvolt, so the readings are given to this figure, though it is not considered as having any significance in temperature measurement.

GAS Thermometer Readings. Expt. 13, May 1, 1903. *Observer, J. A. H.*

Time.	Scale.	Thermometers.				Barometer.	Barometer temperature.
		Dead space below point. (1.)	Columns. (2.)	(3.)	Water circulation. (4.)		
h. m. s.							
5 5 0	816·86	14·7	14·90	14·75	15·2	749·85	14·0
9 0	816·96	·8	·95	·8	·2	·90	·0
14 0	817·00	·95	15·00	·9	·1	·95	·0
17 30	817·00	15·0	·05	·95	·1	750·00	·0
	816·95	14·85	14·95	14·85	15·15	749·92	14·0
Index corr. =	−747·18	− ·00	− ·00	− ·00	− ·40	Scale corr. = − ·05	·1
						Lat. corr. − + ·42	
	+ 69·77	14·85	14·95	14·85	14·75	Temp. corr. − 1·70	13·9
Lat. corr. =	+ ·04						
Temp. corr. =	− ·17					Corr. bar. = 748·59	
	+ 69·64						
	748·59						
H =	818·23						

Mean temperature of dead space = 14·80 = t_1'.

$$H_0\left(1 + \frac{v}{V} \cdot \frac{1}{1 + \alpha t_1}\right) = 184{\cdot}375. \qquad \frac{v}{V} = {\cdot}010_0.$$

$$\alpha = {\cdot}003671_0.$$

$$T_{gas} = 1005{\cdot}2^\circ. \qquad (1 + 3\beta T) = 1{\cdot}01230,$$

where T taken from junction = 1004·3°.

A second approximation, employing 1005·2° in the dilatation term, makes the final gas thermometer temperature 1005·0° C.

THERMOJUNCTION Readings. Expt. 13.

Observer, J. A. H.

Time.	Microvolts.	Cell resistance and temperature.	Series resistance.
h. m. s.			
5 0 0	9570·5 AB	101·89	103·036
2 0	75·8 BA	T = 13·7	—
7 30	76·1 BA	—	—
8 15	73·0 AB	—	—
11 30	73·5 AB	—	103·036
13 0	72·0 AB	—	—
16 0	71·2 AB	—	—
16 30	75·2 BA	—	—
20 40	71·2 AB	—	—
21 0	75·4 BA	—	103·036

From these observations plotted, the following were deduced as the thermojunction readings simultaneous with the gas thermometer readings :—

Time.	Mean microvolts.
h. m. s.	
5 5 0	9573·8
9 0	9574·8
14 0	9573·9
17 30	9573·2
	Mean = 9573·9

From table of E.M.F. of junctions the value of the E.M.F. for 1000° C. = 9524·0, and the difference for 10° = 115·1 microvolts.

Whence the temperature corresponding to 9573·9 = 1004·3°

T (thermojunction) = 1004·3.

PLATINUM Thermometer Readings. Expt. 13.

Observer, W. H.

Time.	Coils.	Bridge wire.	Box temperature.	Centre = −0·29.
h. m. s.				
5 5 0	ABEFGI	−2·530	14·32	—
9 0	—	−2·660	·38	—
14 0	—	−2·686	·44	—
17 30	—	−2·784	·48	—
	1129·454	−2·665	14·40	—

R.	R − R_0.	*pt.*	T ($\delta = 1\cdot50$).	Difference for ·009 in δ.
1125·44	868·79	868·70	1005·18	−·82

$$T(\delta = 1\cdot491) = 1004\cdot37^{\circ}.$$

Summary.—

T gas (found) = 1005·0°
Furnace correction = + 0·0
T gas corrected = 1005·0
T thermojunction = 1004·4
T platinum thermometer = 1004·37

XXIV. *First Set of Comparisons.* October, 1902.

	millims.
Mean ice-point of gas thermometer before observations . . .	336·41
,, ,, ,, after ,, . . .	336·25

Mean value of α before $= \cdot 003675$,
,, ,, after $= \cdot 003676$.

$$\frac{v}{V_0} = \cdot 008_3.$$

Platinum thermometer BA_2 :—

$R_0 = 2\cdot 57453$ ohms before commencing,
$= 2\cdot 57465$,, after seventh experiment,
$= 2\cdot 57433$,, at conclusion,

F.I. $= 1\cdot 00034$ before commencing,
$= 1\cdot 00020$ at end,

$\delta = 1\cdot 510$ mean of all observations.

FIRST Set of Comparisons. Furnace wound uniformly. BA_2 and Gas Thermometer.

No. of experiment.	*pt.*	T ($\delta = 1\cdot 50$) from Table I.	T ($\delta = 1\cdot 491$) from Table II.	Gas thermometer reading.	Compensation correction.	Corrected gas thermometer reading.
1	482·54	514·66	514·47	510·2	+5	515
2	604·96	660·48	660·15	655·2	+4	659
3	606·48	662·35	662·02	659·1	+4	663
4	610·02	666·69	666·36	661·7	+4	665½
5	459·21	487·66	487·49	483·4	+5	488½
6	749·07	843·03	842·46	839·7	+3	842½
7	750·24	844·56	843·99	840·7	+3	843½
8	481·53	513·36	513·17	509·5	+5	514½
9	394·96	414·52	414·40	411·2	+5½	416½

RESULTS of Comparison arranged in Order of Ascending Temperature.

	T from platinum.	T from gas corrected.	Difference. T gas – T platinum.
9	414·4	416½	+2
5	487·5	488½	+1
8	513·2	514½	+1
1	514·5	515	+ ·5
2	660·1	659	–1
3	662·0	663	+1
4	666·4	665½	–1
6	842·5	842½	+0
7	844·0	843½	– ·5

Second Set of Comparisons. April and May, 1903.

	millims.
Mean ice-point of gas thermometer before comparisons	$180{\cdot}92_3$
,, ,, ,, after ,, ...	$182{\cdot}66$

Mean value of α from determinations before comparisons = ·003670,
,, ,, ,, after ,, = ·003671.

$$\frac{v}{V_0} = \cdot 010_0.$$

In this case there is a change in the zero-point greater than that observed in the first set and in the opposite direction. In allowing for this, the change was supposed to have been proportional to the time.

Platinum thermometer BA_2 :—

R_0 = 2·57251 ohms before commencing comparisons,
= 2·57198 ,, at conclusion of ,,

F.I. = 1·00008 before commencing ,,
= 1·00000 at conclusion of ,,

δ = 1·491 mean of large number of observations extending over six months.

The platinum thermometer constants of this set are expressed in terms of a slightly different unit from that employed in first set six months earlier.

SECOND Series of Comparisons; Compensated Furnace.

Gas Thermometer, Platinum Thermometer BA_2, and Thermojunction NPL 2.

The comparisons are for the sake of reference arranged in order of ascending temperature.

No. of experiment.	Gas thermometer.			Thermojunction.		Platinum thermometer.				Differences.	
	Reading of gas thermometer.	Furnace correction.	Corrected temperature from gas thermometer.	Micro-volts of junction.	Temperature from junction.	*pt*.	T, CALLENDAR'S formula, $\delta = 1 \cdot 50$ from Table I.	Temperature from platinum thermometer, $\delta = 1 \cdot 491$ from Table II.	Gas–platinum.	Gas thermometer.	Platinum thermometer.
5	521·4	+1·7	523·1	4434	524·3	491·18	524·59	524·39	−1·3	−1·2	+·1
10	566·9	+1·6	568·5	4885	569·5	529·44	569·59	569·35	−0·8	−1·0	−·2
8	597·0	+1·5	598·5	5172	597·8	553·22	597·89	597·62	+ ·9	+0·7	−·2
1	597·9	+1·5	599·4	5183	599·0	554·21	599·07	598·80	+ ·6	+0·4	−·2
3	639·7	+1·4	641·1	5614	641·1	589·03	641·06	640·75	+ ·4	+0·0	−·4
4	681·1	+1·3	682·4	6048	683·0	623·20	682·90	682·54	− ·1	−0·6	−·5
2	775·6	+1·1	776·7	7028	775·5	696·97	775·60	775·13	+1·6	+1·2	−·4
9	819·0	+1·0	820·0	7492	818·4	730·55	818·84	818·31	+1·7	+1·6	−·1
11	830·4	+1·0	831·4	7643	832·2	740·94	832·41	831·86	− ·5	−0·8	−·3
12	865·8	+ ·9	866·7	8040	868·4	768·68	868·89	868·28	−1·6	−1·7	−·1
6	874·2	+ ·8	875·0	8117	875·4	773·92	875·86	875·24	− ·2	−0·4	−·2
7	959·5	+ ·3	959·8	9022	956·0	833·40	956·21	955·47	+4·3	+3·8	−·5
13	1005·0	+ ·0	1005·0	9574	1004·4	868·70	1005·19	1004·37	+ ·6	+0·6	−·0

Third Set of Comparisons. May and June, 1903.

	millims.
Mean ice-point of gas thermometer before comparison	= 176·28
,, ,, ,, after ,,	= 176·19

Mean value of α from determinations before = ·003660
,, ,, ,, after = ·003661

Platinum thermometer K_2 :—

R_0 = 2·62864 ohms before commencing comparisons,
= 2·62705 ,, at conclusion of ,,

F.I. = 1·00789 before commencing ,,
= 1·00819 at conclusion of ,,

δ = 1·510 mean value over a long period.

THIRD Series of Comparisons in Compensated Furnace.

Gas Thermometer, Platinum Thermometer K_2, and Thermojunction NPL 2.

No. of Experiment.	Gas thermometer.			Thermojunction.		Platinum thermometer.			Differences.		
	Reading of gas thermometer.	Furnace correction.	Corrected temperature from gas thermometer.	Micro-volts of junction.	Temperature from junction.	*pt.*	T, $\delta = 1 \cdot 50$ from Table I.	T, $\delta = 1 \cdot 51$ from Table II.	Gas–platinum.	Gas thermometer.	Platinum thermometer.
2	451·8	+1·9	453·7	3750	454·4	429·30	453·33	453·49	+ ·2	− ·7	− ·9
6	462·7	+1·9	464·6	3862	465·9	438·98	464·36	464·53	+ ·1	−1·3	−1·4
1	537·6	+1·7	539·3	4597	540·7	503·92	539·48	539·72	− ·4	−1·4	−1·0
3	559·2	+1·6	560·8	4814	562·5	522·14	560·93	561·19	+ ·4	−1·7	−1·3
5	641·2	+1·4	642·6	5648	644·4	590·21	642·49	642·83	− ·2	−1·8	−1·6
4	675·5	+1·4	676·9	6001	678·5	618·04	676·56	676·94	+ ·0	−1·6	−1·6
7	752·6	+1·2	753·8	6829	756·9	680·47	754·56	755·05	−1·2	−3·1	−1·9
9	844·1	+ ·9	845·0	7779	844·6	748·18	841·87	842·49	+2·5	+ ·4	−2·1
8	864·6	+ ·9	865·5	8011	865·6	764·39	863·23	863·89	+1·6	− ·1	−1·7
11	936·0	+ ·4	936·4	8757	932·5	813·70	929·30	930·07	+6·3	+3·9	−2·4
10	969·8	+ ·2	970·0	9165	968·6	839·78	964·98	965·82	+4·2	+1·4	−2·8

XXV. *Conclusions.*

These three sets of experiments show that :—

(1.) The readings of platinum thermometers BA_2 and K_2, which may be taken as representative samples of pure platinum, when reduced to the air-scale by CALLENDAR'S formula, employing his value for the boiling-point of sulphur, are in reasonably close agreement with the results given by the constant-volume nitrogen thermometer employing chemical nitrogen under low initial pressure and using the revised values for the dilatation of porcelain. The divergence of the two scales only exceeds the probable error at the higher part of the range.

(2.) The platinum thermometers, and the thermojunctions representing the temperature scale of the Reichsanstalt, based on measurements made with the gas thermometer with bulb of platinum iridium, agree very closely, the thermojunction giving apparently a slightly higher value throughout the range covered by the experiments.

As the result of these comparisons is to justify the application of CALLENDAR'S parabolic formula up to 1000° C., the tables previously alluded to for deducing the value of T from pt over the whole range for which platinum thermometers are useful, $-200°$ to $+1100°$, are given. From $-200°$ to $+200°$ the values are given for each degree,[*] and from 200° to 1100° for each 10°.

For the figures given in Table I., in their final form, calculated by a method which renders unlikely the accumulation of errors greater than 1 unit in the last figure given, I am indebted to Mr. F. J. SELBY.

I have also to acknowledge my indebtedness to Mr. W. HUGO, who assisted during the comparisons by taking the observations with the platinum thermometers, and especially to the Director of the Laboratory, Dr. GLAZEBROOK, for his continued interest and advice on many points.

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[*] In the paper as printed this table has been shortened by only giving the value of pt for intervals of 1° over the range $-50°$ to $+150°$.

XXVI. Table I.

pt.	T.	Difference for 1 *pt.*	Proportional parts.		*pt.*	T.	Difference for 1 *pt.*	Proportional parts.	
−200	−191·62	·934			− 39	− 38·208	·974		
−190	−182·28	·936			− 38	− 37·234	·975		
−180	−172·92	·939			− 37	− 36·259	·975		
−170	−163·53	·941			− 36	− 35·284	·975		
−160	−154·12	·943			− 35	− 34·309	·975		
−150	−144·69	·946			− 34	− 33·334	·976		
−140	−135·23	·949			− 33	− 32·358	·976		
−130	−125·74	·951			− 32	− 31·382	·976		
−120	−116·23	·954			− 31	− 30·406	·977		
−110	−106·69	·956			− 30	− 29·429	·977		
−100	− 97·13	·959			− 29	− 28·452	·977		
− 90	− 87·54	·962			− 28	− 27·475	·978		
− 80	− 77·92	·964			− 27	− 26·497	·978		
− 70	− 68·28	·967			− 26	− 25·519	·978		
− 60	− 58·61	·970			− 25	− 24·541	·978		
− 50	− 48·91	·971			− 24	− 23·563	·979		
− 49	− 47·937	·972	970		− 23	− 22·584	·979		
− 48	− 46·965	·972			− 22	− 21·605	·979		
− 47	− 45·993	·972	·1	·097	− 21	− 20·626	·979		
− 46	− 45·021	·973	·2	·194	− 20	− 19·647	·980		
− 45	− 44·048	·973	·3	·291	− 19	− 18·667	·980	980	
− 44	− 43·075	·973	·4	·388	− 18	− 17·687	·980		
− 43	42·102	·973	·5	·485	− 17	− 16·707	·980	·1	·098
− 42	41·129	·973	·6	·582	− 16	− 15·727	·981	·2	·196
− 41	− 40·156	·974	·7	·679	− 15	− 14·746	·981	·3	·294
− 40	− 39·182	·974	·8	·776	− 14	− 13·765	·981	·4	·392
− 39	− 38·208	·974	·9	·873	− 13	− 12·784	·982	·5	·490

TABLE I.—continued.

pt.	T.	Difference for 1° *pt.*	Proportional parts.		*pt.*	T.	Difference for 1° *pt.*	Proportional parts.	
– 13	– 12·784	·982			+ 13	+ 12·832	·989		
– 12	– 11·802	·982	·6	·588	+ 14	+ 13·821	·989		
– 11	– 10·820	·982	·7	·686	+ 15	+ 14·811	·989		
– 10	– 9·838	·983	·8	·784	+ 16	+ 15·800	·990		
– 9	– 8·855	·983	·9	·882	+ 17	+ 16·790	·991	990	
– 8	– 7·872	·983			+ 18	+ 17·781	·990		
– 7	– 6·889	·983			+ 19	+ 18·771	·991	·1	·099
– 6	– 5·906	·984			+ 20	+ 19·762	·991	·2	·198
– 5	– 4·922	·984			+ 21	+ 20·753	·992	·3	·297
– 4	– 3·938	·984			+ 22	+ 21·745	·992	·4	·396
– 3	– 2·954	·984			+ 23	+ 22·737	·992	·5	·495
– 2	– 1·970	·985			+ 24	+ 23·729	·992	·6	·594
– 1	– 0·985	·985			+ 25	+ 24·721	·993	·7	·693
+ 0	+ 0·000	·985			+ 26	+ 25·714	·993	·8	·792
+ 1	+ 0·985	·986			+ 27	+ 26·707	·993	·9	·891
+ 2	+ 1·971	·986			+ 28	+ 27·700	·993		
+ 3	+ 2·957	·986			+ 29	+ 28·693	·994		
+ 4	+ 3·943	·987			+ 30	+ 29·687	·994		
+ 5	+ 4·930	·987			+ 31	+ 30·681	·994		
+ 6	+ 5·917	·987			+ 32	+ 31·675	·995		
+ 7	+ 6·904	·987			+ 33	+ 32·670	·995		
+ 8	+ 7·891	·988			+ 34	+ 33·665	·995		
+ 9	+ 8·879	·988			+ 35	+ 34·660	·996		
+ 10	+ 9·867	·988			+ 36	+ 35·656	·996		
+ 11	+ 10·855	·988			+ 37	+ 36·652	·996		
+ 12	+ 11·843	·988			+ 38	+ 37·648	·996		
+ 13	+ 12·832	·989			+ 39	+ 38·644	·997		

TABLE I.—continued.

pt.	T.	Difference for 1° *pt.*	Proportional parts.		*pt.*	T.	Difference for 1° *pt.*	Proportional parts.	
+ 39	+ 38·664				+ 65	+ 64·657			
		·997					1·005		
+ 40	+ 39·641				+ 66	+ 65·662			
		·997					1·005		
+ 41	+ 40·638				+ 67	+ 66·667			
		·997					1·005		
+ 42	+ 41·635				+ 68	+ 67·672			
		·998					1·005		
+ 43	+ 42·633				+ 69	+ 68·677			
		·998					1·006		
+ 44	+ 43·631				+ 70	+ 69·683			
		·998					1·006		
+ 45	+ 44·629				+ 71	+ 70·689			
		·999					1·007		
+ 46	+ 45·628				+ 72	+ 71·696			
		·999					1·007		
+ 47	+ 46·627		1000		+ 73	+ 72·703			
		·999					1·007		
+ 48	+ 47·626				+ 74	+ 73·710			
		·999					1·007		
+ 49	+ 48·625		·1	·100	+ 75	+ 74·717			
		1·000					1·008		
+ 50	+ 49·625		·2	·200	+ 76	+ 75·725			
		1·000					1·008		
+ 51	+ 50·625		·3	·300	+ 77	+ 76·733			
		1·000					1·008		
+ 52	+ 51·625		·4	·400	+ 78	+ 77·741			
		1·001					1·008		
+ 53	+ 52·625		·5	·500	+ 79	+ 78·749			
		1·001					1·009		
+ 54	+ 53·627		·6	·600	+ 80	+ 79·758			
		1·001					1·009		
+ 55	+ 54·628		·7	·700	+ 81	+ 80·767		1010	
		1·002					1·009		
+ 56	+ 55·630		·8	·800	+ 82	+ 81·776			
		1·002					1·010		
+ 57	+ 56·632		·9	·900	+ 83	+ 82·786		·1	·101
		1·002					1·010		
+ 58	+ 57·634				+ 84	+ 83·796		·2	·202
		1·002					1·011		
+ 59	+ 58·636				+ 85	+ 84·807		·3	·303
		1·003					1·010		
+ 60	+ 59·639				+ 86	+ 85·817		·4	·404
		1·003					1·011		
+ 61	+ 60·642				+ 87	+ 86·828		·5	·505
		1·003					1·011		
+ 62	+ 61·645				+ 88	+ 87·839		·6	·606
		1·004					1·012		
+ 63	+ 62·649				+ 89	+ 88·851		·7	·707
		1·004					1·012		
+ 64	+ 63·653				+ 90	+ 89·863		·8	·808
		1·004					1·012		
+ 65	+ 64·657				+ 91	+ 90·875		·9	·909
		1·005					1·013		

Table I.—continued.

pt.	T.	Difference for 1 *pt*.	Proportional parts.		*pt*.	T.	Difference for 1 *pt*.	Proportional parts.	
+ 91	+ 90·875	1·013			+117	+117·304	1·021		
+ 92	+ 91·888	1·013			+118	+118·325	1·021	·6	·612
+ 93	+ 92·901	1·013			+119	+119·346	1·022	·7	·714
+ 94	+ 93·914	1·014			+120	+120·368	1·021	·8	·816
+ 95	+ 94·928	1·014			+121	+121·389	1·022	·9	·918
+ 96	+ 95·942	1·014			+122	+122·411	1·023		
+ 97	+ 96·956	1·014			+123	+123·434	1·023		
+ 98	+ 97·970	1·015			+124	+124·457	1·023		
+ 99	+ 98·985	1·015			+125	+124·480	1·023		
+100	+100·000	1·015			+126	+126·503	1·023		
+101	+101·015	1·016			+127	+127·526	1·024		
+102	+102·031	1·016			+128	+128·550	1·025		
+103	+103·047	1·016			+129	+129·575	1·024		
+104	+104·063	1·017			+130	+130·599	1·025		
+105	+105·080	1·017			+131	+131·624	1·026		
+106	+106·097	1·017			+132	+132·650	1·025		
+107	+107·114	1·018			+133	+133·675	1·026		
+108	+108·132	1·018			+134	+134·701	1·026		
+109	+109·150	1·018			+135	+135·727	1·027		
+110	+110·168	1·018			+136	+136·754	1·027		
+111	+111·186	1·019	1020		+137	+137·781	1·027		
+112	+112·205	1·019			+138	+138·808	1·028	1030	
+113	+113·224	1·020	·1	·102	+139	+139·836	1·027		
+114	+114·244	1·020	·2	·204	+140	+140·863	1·028	·1	·103
+115	+115·264	1·020	·3	·306	+141	+141·891	1·029	·2	·206
+116	+116·284	1·020	·4	·408	+142	+142·920	1·029	·3	·309
+117	+117·304	1·021	·5	·510	+143	+143·949	1·029	·4	412

TABLE I.—continued.

pt.	T.	Difference for 1° *pt.*	Proportional parts.		*pt.*	T.	Difference for 1° *pt.*	Proportional parts.
+143	+143·949	1·029			+340	+353·44	1·102	
+144	+144·978	1·030	·5	·515	+350	+364·46	1·106	
+145	+146·008	1·029	·6	·618	+360	+375·52	1·110	
+146	+147·037	1·030	·7	·721	+370	+386·62	1·114	
+147	+148·067	1·031	·8	·824	+380	+397·77	1·118	
+148	+149·098	1·031	·9	·927	+390	+408·95	1·123	
+149	+150·129	1·031			+400	+420·18	1·127	
+150	+151·16	1·033			+410	+431·45	1·132	
+160	+161·49	1·036			+420	+442·77	1·136	
+170	+171·85	1·040			+430	+454·13	1·140	
+180	+182·25	1·043			+440	+465·53	1·140	
+190	+192·68	1·046			+450	+476·97	1·149	
+200	+203·14	1·051			+460	+488·46	1·154	
+210	+213·65	1·053			+470	+500·00	1·158	
+220	+224·18	1·057			+480	+511·58	1·163	
+230	+234·75	1·060			+490	+523·21	1·168	
+240	+245·35	1·064			+500	+534·89	1·173	
+250	+255·99	1·068			+510	+546·62	1·178	
+260	+266·67	1·071			+520	+558·40	1·182	
+270	+277·38	1·075			+530	+570·22	1·188	
+280	+288·13	1·079			+540	+582·10	1·193	
+290	+298·92	1·083			+550	+594·03	1·198	
+300	+309·75	1·086			+560	+606·00	1·203	
+310	+320·61	1·090			+570	+618·03	1·208	
+320	+331·51	1·094			+580	+630·11	1·213	
+330	+342·46	1·098			+590	+642·24	1·219	
+340	+353·44	1·102			+600	+654·43	1·224	

Table I.—continued.

pt.	T.	Difference for 1° *pt.*	Proportional parts.	*pt.*	T.	Difference for 1° *pt.*	Proportional parts.
+600	+654·43			+ 850	+ 979·11		
		1·224				1·390	
+610	+666·67			+ 860	+ 993·01		
		1·230				1·399	
+620	+678·97			+ 870	+1007·00		
		1·235				1·407	
+630	+691·32			+ 880	+1021·07		
		1·241				1·416	
+640	+703·73			+ 890	+1035·23		
		1·247				1·424	
+650	+716·20			+ 900	+1049·47		
		1·253				1·433	
+660	+728·73			+ 910	+1063·80		
		1·259				1·441	
+670	+741·32			+ 920	+1078·21		
		1·264				1·450	
+680	+753·96			+ 930	+1092·71		
		1·271				1·460	
+690	+766·67			+ 940	+1107·31		
		1·277				1·469	
+700	+779·44			+ 950	+1122·00		
		1·283				1·479	
+710	+792·27			+ 960	+1136·79		
		1·290				1·490	
+720	+805·17			+ 970	+1151·69		
		1·296				1·499	
+730	+818·13			+ 980	+1166·68		
		1·303				1·508	
+740	+831·16			+ 990	+1181·76		
		1·310				1·519	
+750	+844·26			+1000	+1196·95		
		1·316				1·529	
+760	+857·42			+1010	+1212·24		
		1·323				1·541	
+770	+870·65			+1020	+1227·65		
		1·330				1·552	
+780	+883·95			+1030	+1243·17		
		1·337				1·563	
+790	+897·32			+1040	+1258·80		
		1·344				1·575	
+800	+910·76			+1050	+1274·55		
		1·352				1·587	
+810	+924·28			+1060	+1290·42		
		1·359				1·599	
+820	+937·87			+1070	+1306·41		
		1·367				1·611	
+830	+951·54			+1080	+1322·52		
		1·374				1·624	
+840	+965·28			+1090	+1338·76		
		1·383				1·637	
+850	+979·11			+1100	+1355·13		
		1·390					

XXVII. TABLE II.—*To Calculate the Change in* T *for a Given Small Change in* δ.

T.	Change in T for change of + ·01 in δ.	T.	Change in T for change of + ·01 in δ.
−200	+ ·0600	+ 250	+ ·0375
−180	+ ·0504	+ 300	+ ·0600
−160	+ ·0416	+ 350	+ ·0875
−140	+ ·0336	+ 400	+ ·1200
−120	+ ·0264	+ 450	+ ·1575
−100	+ ·0200	+ 500	+ ·200
− 80	+ ·0144	+ 550	+ ·247
− 60	+ ·0096	+ 600	+ ·300
− 40	+ ·0056	+ 650	+ ·357
− 20	+ ·0024	+ 700	+ ·420
− 0	+ ·0000	+ 750	+ ·487
+ 20	− ·0016	+ 800	+ ·560
+ 40	− ·0024	+ 850	+ ·637
+ 60	− ·0024	+ 900	+ ·720
+ 80	− ·0016	+ 950	+ ·807
+100	− ·0000	+1000	+ ·900
+120	+ ·0024	+1050	+ ·997
+140	+ ·0056	+1100	+1·100
+160	+ ·0096	+1150	+1·207
+180	+ ·0144	+1200	+1·320
+200	+ ·0200	+1250	+1·437

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er. It is also shown
glass the ruby colour
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form in a glass what

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XXVII. Table II.—*To Calculate the Change in* T *for a Given Small*

T.	C
−200	
−180	
−160	
−140	
−120	
−100	
− 80	
− 60	
− 40	
− 20	
− 0	
+ 20	
+ 40	
+ 60	
+ 80	
+100	
+120	
+140	
+160	
+180	
+200	

XII. *Colours in Metal Glasses and in Metallic Films.*

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Communicated by Professor J. Larmor, *Sec. R.S.*

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Introduction.

§ 1. The present paper contains a discussion of some optical properties of a medium containing minute metal spheres. The discussion is divided into two Parts: the first Part dealing with colours in metal glasses, in which the proportion of volume occupied by metal is small; the second Part dealing with metal films, in which this proportion may have any value from zero to unity.

In Part I. the observations of Siedentopf and Zsigmondy beyond the limit of microscopic vision ('Ann. der Phys.,' January, 1903) are discussed. It is shown that the particles seen in a gold ruby glass are particles of gold which, when their diameters are less than $0{\cdot}1\mu$, are accurately spherical. I have endeavoured to show that the presence of many of these minute spheres to a wave-length of light in the glass will account for all the optical properties of "regular" gold ruby glass, and that the irregularities in colour and in polarisation effects sometimes exhibited by gold glass are due to excessive distance between consecutive gold particles or to excessive size of such particles, the latter, however, involving the former. It is also shown that the radiation from radium is capable of producing in gold glass the ruby colour which is generally produced by re-heating. The method adopted enables us to predict from a knowledge of the metal present in metallic form in a glass what colour that glass will be in its "regular" state.

In Part II. the optical properties, and the changes in colour on heating, of the silver and gold films observed by Mr. G. T. Beilby ('Roy. Soc. Proc.,' vol. 72, p. 226), and of the potassium and sodium films deposited on glass by Professor R. W. Wood ('Phil. Mag.,' p. 396, 1902), are discussed, with a view to showing that they can be accounted for by supposing the films to be composed of minute metal spheres of varying sizes.

PART I.

§ 2. Consider the incidence of light of wave-length λ on a sphere ot metal of radius a. Suppose the constants of the metal relative to the surrounding medium, which we may first suppose to be æther, are n, the coefficient of refraction, and κ, the coefficient of absorption. Let us write

$$N \equiv n(1 - \iota\kappa) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where, as usual, ι denotes $\sqrt{-1}$.

We shall use the following notation to denote the electric vector :—

Incident light . . . . $\mathbf{E}_0\,\{X_0 = \exp\{\iota p(t - z/c)\},\ Y_0 = 0,\ Z_0 = 0\}$.

Transmitted + reflected light . . $\mathbf{E}_1\,\{X_1,\ Y_1,\ Z_1\}$.

Here $p = 2\pi c/\lambda$, c being the velocity of light *in vacuo*.

HERTZ ('Ausbreitung der electrischen Kraft,' Leipzig, 1892, p. 150) has shown that the electric and magnetic forces at any point (x, y, z) due to an oscillating electric doublet of moment $Ae^{\iota pt}$ along the axis of x are given by

$$\mathbf{E} = \nabla\frac{\partial\Pi}{\partial x} - (\nabla^2\Pi,\ 0,\ 0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$\mathbf{H} = \frac{1}{c}\left(0,\ \frac{\partial^2\Pi}{\partial z\,\partial t},\ -\frac{\partial^2\Pi}{\partial y\,\partial t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where

$$\Pi = A/r.\exp\{\iota p(t - r/c)\},$$

for these expressions satisfy MAXWELL'S equations

$$\frac{d\mathbf{E}}{dt} = c\ \mathrm{curl}\ \mathbf{H},\quad \frac{d\mathbf{H}}{dt} = -c\ \mathrm{curl}\ \mathbf{E}\quad \text{and}\quad \mathrm{div}\ \mathbf{E} = \mathrm{div}\ \mathbf{H} = 0,$$

and when r is very small compared with the wave-length ($\lambda = 2\pi c/p$) of the emitted waves the expression for $\mathbf{E}$ reduces to

$$\mathbf{E} = \nabla(\partial\Pi/\partial x),$$

which is at any time the electric force which would be electrostatically due to the doublet if its moment remained constant and equal to its value at that time.

Lord RAYLEIGH ('Phil. Mag.,' XLIV., pp. 28–52, 1897, and 'Collected Papers,' vol. 4, p. 321) has extended this theorem to the case of a very small sphere. In the region for which the distance, r, from the centre of a small sphere of radius a excited by an electric field $\mathbf{E} = (e^{\iota pt}, 0, 0)$, is small compared with the wave-length, the electric force due to the sphere is

$$\mathbf{E}_1 = \nabla \frac{\partial}{\partial x}\left(\frac{\mathrm{K}-1}{\mathrm{K}+2}\frac{a^3}{r}\right).\, e^{\iota pt}.$$

By comparing this with HERTZ's corresponding result

$$\mathbf{E} = \nabla \frac{\partial}{\partial x}\left(\frac{e^{\iota pt}}{r}\right)$$

for an oscillating doublet of moment $e^{\iota pt}$, as given above, it appears from (2) and (3) that the electric and magnetic forces at any point, due to waves emitted by the sphere, must be given by the equations

$$\mathbf{E}_1 = \nabla \frac{\partial \Pi}{\partial x} - (\nabla^2\Pi,\ 0,\ 0), \qquad \mathbf{H}_1 = \frac{1}{c}\left(0,\ \frac{\partial^2\Pi}{\partial x\,\partial t},\ -\frac{\partial^2\Pi}{\partial y\,\partial t}\right) \quad . \quad . \quad (4,\ 5),$$

where now

$$\Pi = \frac{\mathrm{K}-1}{\mathrm{K}+2}\cdot\frac{a^3}{r}\cdot \exp\{\iota p\,(t - r/c)\}.$$

Replacing K by N^2, where N is the quantity defined by equation (1), we conclude that when a metal sphere is excited by a periodic electric force $\mathbf{E}_0$, it emits the waves which would be emitted by a Hertzian doublet which at time t was of moment equal to

$$\frac{\mathrm{N}^2-1}{\mathrm{N}^2+2}\,a^3\mathbf{E}_0.$$

The same result can be proved directly by adapting the analysis given by L. LORENZ ('Vidensk. Selsk. Skr.,' Copenhagen, 1890) to the electromagnetic theory. The problem has also been treated by STOKES ('Camb. Trans.,' vol. 9, p. 1, 1849, and 'Papers,' vol. 4, p. 245, p. 262).

At a great distance from the origin, *i.e.*, when r is great compared with λ, equation (4) reduces to [*cf.* RAYLEIGH, *loc. cit.*, equation (106)]

$$\mathbf{E}_1 = \frac{4\pi^2 a^3}{\lambda^2 r}\,\frac{\mathrm{N}^2-1}{\mathrm{N}^2+2}\exp\{\iota p\,(t-r/c)\}\left\{\frac{y^2+z^2}{r^2},\ -\frac{xy}{r^2},\ -\frac{xz}{r^2}\right\} \quad . \quad . \quad (6).$$

If we transform to spherical co-ordinates $\bar{\mathrm{X}}$, $\bar{\mathrm{Y}}$, $\bar{\mathrm{Z}}$ in the respective directions of increase of r, θ, ϕ (fig. 1) we obtain, at a great distance from the origin,

$$\bar{\mathrm{X}}_1 = 0,\ \ \bar{\mathrm{Y}}_1 = \frac{4\pi^2 a^3}{\lambda^2 r}\,\frac{\mathrm{N}^2-1}{\mathrm{N}^2+2}\cdot \exp\{\iota p\,(t-r/c)\}\cos\theta\cos\phi,$$

$$\bar{\mathrm{Z}}_1 = -\frac{4\pi^2 a^3}{\lambda^2 r}\,\frac{\mathrm{N}^2-1}{\mathrm{N}^2+2}\exp\{\iota p\,(t-r/c)\}\sin\phi. \quad . \quad . \quad . \quad . \quad . \quad (7).$$

It appears from equations (6) or (7) that such a small sphere, in common with any other minute system whose moment is proportional to the electric vector of the incident light, emits light with an intensity proportional to the inverse fourth power of the wave-length, provided that N is independent of λ. It is this property which, as Lord RAYLEIGH has shown, accounts for the blue colour of the light received from the sky.

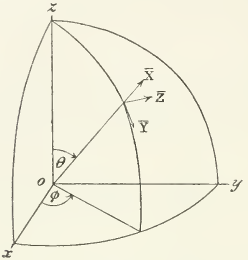

Fig. 1.

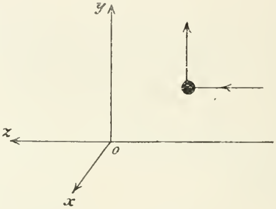

Fig. 2.

§ 3. In the 'Annalen der Physik' for January, 1903, H. SIEDENTOPF and R. ZSIGMONDY publish some observations on the metal particles in gold ruby glasses. By their method of illumination they were able to see particles whose dimensions were of the order of from 4 to 7 $\mu\mu$, where $\mu\mu$ represents 10^{-6} millim.

The arrangement consisted of a system of lenses following a strongly illuminated and very narrow slit. The system of lenses, of which the last is a low power microscopic objective, serves as a condenser and forms a very narrow image of the slit inside the glass under observation. This image of the slit may not be more than one or two wave-lengths thick.

The observation is made with a microscope having the tube perpendicular to the incident light, so that only the light emitted by the metallic particles travels up the tube. This is the light the electric vector of which has been distinguished by the suffix unity in the preceding analysis. The image of the slit, which is parallel to $0x$ in fig. 2, comes directly under the microscope tube, which is in the direction $0y$; thus only the particles illuminated at the image of the slit send light up the tube. The diffraction discs do not pile up on top of one another if the average distance between two metal particles is greater than the thickness of the image of the slit. In this case, then, the number of particles per unit area can be counted.

On pp. 11 and 12 of the paper referred to, SIEDENTOPF and ZSIGMONDY discuss the appearances in the second focal plane of the microscope when the light incident in the glass is plane polarised. The figs. 3–6 above are reproduced from their paper. In fig. 3 the plane of polarisation of the incident light was that of incidence, the plane of incidence being the plane containing the axis of the microscope and that of the incident pencil of light; in figs. 4 and 5 the plane of polarisation of the incident light was inclined at 45° to the plane of incidence; while in fig. 6 the two planes were perpendicular.

In the figs. 3–6 the upper diagram represents the second focal plane of the microscope when the diameters of the particles of metal in the glass are less than 0·1 μ, the small lines being parallel to the planes of polarisation of the emergent light in various parts of the field, the "emergent light" here meaning the light sent up the microscope tube by the metal particles in the glass under observation. The lower diagrams in the same figures represent the appearances of a diffraction disc for the same respective positions of the plane of polarisation of the incident light.

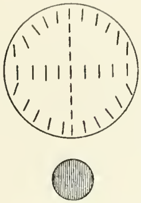
Fig. 3.

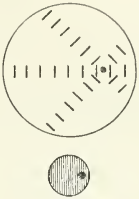
Fig. 4.

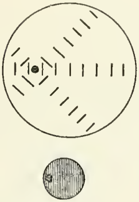
Fig. 5.

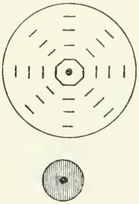
Fig. 6.

It is to be noticed that the light emitted in any particular direction comes to a focus at a corresponding point in the second focal plane of the microscope. Consequently a black spot in that plane means that no light is emitted in the corresponding direction.

If all the particles are spheres sending up no light in some particular direction, there will thus be a black spot in the second focal plane, as well as in each diffraction disc, at the point corresponding to that direction.

Suppose now, as in § 2, that the incident light travels in the direction $0z$ and is polarised in plane $y0z$, fig. 1. Instead of conceiving this plane to alter as we consider the various cases of figs. 3–6, we shall imagine the microscope tube to move in the plane $x0y$.

Thus in fig. 3 the microscope is along $0y$, in fig. 6 along $0x$, while in figs. 5 and 4 the tube lies in the intermediate positions, namely, $\theta = 90°$, $\phi = \pm 45°$ respectively.

It will now be shown that the figs. 3–6 are completely accounted for if the particles are spheres small compared with a wave-length, *i.e.*, appreciably smaller than $0{\cdot}1\ \mu$.

From equations (7) the character of the light emitted by such a sphere in the direction θ, ϕ (fig. 1) is determined by the electric force $\mathbf{E}_1$ whose composition is:

$$\bar{X}_1 = 0, \quad \bar{Y}_1 = B\cos\theta\cos\phi, \quad \bar{Z}_1 = -B\sin\phi \quad . \quad . \quad . \quad . \quad . \quad (8),$$

where

$$B = \frac{4\pi^2 a^3}{\lambda^2 r} \cdot \frac{N^2 - 1}{N^2 + 2} \cdot \exp\{\iota p(t - r/c)\}.$$

Suppose first that, as in the case corresponding to fig. 3, the microscope tube is along $0y$ (fig. 1), the centre of the field then corresponds to $\theta = 90°$, $\phi = 90°$.

The fig. 7 represents the direction of $\mathbf{E}_1$, as deduced from equation (8), for positions, the co-ordinates of which are θ, ϕ, the centre of the diagram corresponding to $\theta = \phi = 90°$, the axis of y. The same figure will therefore represent the directions of the electric vector in various parts of the second focal plane of the microscope.

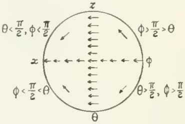
z
$\theta < \frac{\pi}{2}, \phi < \frac{\pi}{2}$
$\phi > \frac{\pi}{2} > \theta$
x
ϕ
$\phi < \frac{\pi}{2} < \theta$
$\theta > \frac{\pi}{2}, \phi > \frac{\pi}{2}$
θ

Fig. 7.

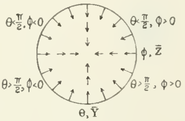
$\theta < \frac{\pi}{2}, \phi < 0$
$\theta < \frac{\pi}{2}, \phi > 0$
$\phi, \bar{Z}$
$\theta > \frac{\pi}{2}, \phi < 0$
$\theta > \frac{\pi}{2}, \phi > 0$
$\theta, \bar{Y}$

Fig. 8.

From equation (8) it appears that when either $\theta = 90°$ or $\phi = 90°$, we shall have $\bar{Y}_1 = 0$, and therefore $\mathbf{E}_1$ becomes $(0, 0, Z_1)$ and only has a component in the direction ϕ. This is represented by the arrows for positions on the axes in fig. 7.

In the middle of the quadrants the directions of the electric vector are no longer parallel to the axis of ϕ but are tilted as in the figure, being tilted in the same manner in opposite quadrants.

Now the planes of polarisation are perpendicular to the electric vector, and the small lines in fig. 3 are perpendicular to the arrows in fig. 7. When, therefore, the incident light is polarised in the plane of incidence, the appearances are accounted for if the particles are small spheres.

Next consider the case corresponding to fig. 6, when the microscope tube is above $0x$. The centre of the field is then $\theta = 90°$, $\phi = 0$. The arrows represent the direction of $\mathbf{E}_1$ in various parts of the field. All these arrows point nearly towards the centre. Along the two axes they point accurately towards the centre. There is no force at the centre, for then both $\bar{Y}_1$ and Z_1 vanish. Consequently, a black spot should appear at the centre, if the particles were spheres. Finally, lines perpendicular to the arrows in fig. 8 are parallel to the lines in fig. 4. Consequently, in this case also the appearances are explained by supposing the particles to be spheres.

In this case, namely, when the incident light is polarised perpendicular to the plane of incidence, it further appears that if an analysing nicol be introduced so as to polarise the emergent light in the plane of incidence, then the analysing nicol removes the Y_1 component of E_1 and the vanishing of Z_1 also, for $\phi = 0$ causes a dark band to cross the field over the diffraction disc if there be only one particle sending light up the tube, the dark band lying along the axis of θ in fig. 8, *i.e.*, in the plane of incidence, and this also was observed by SIEDENTOPF and ZSIGMONDY for the particles in gold glass (*loc. cit.*, p. 12).

The discussion of the cases of figs. 4 and 5 presents no difficulty. The phenomena, including the correct position of the black spot, are again explained, by means of the hypothesis that the small particles are spheres.

Thus all the phenomena, observed in the second focal plane of the microscope, due to particles smaller than $0{\cdot}1\mu$, are exactly those which would be produced by spheres of metal of radius small compared with the length of a wave of light in the glass.

If now the particles were small spheroids, or crystalline in structure, then the position of the black spots, if indeed any existed, and the positions of the plane of polarisation of the light emitted from the particles, would depend on the orientation of the particles. Unless, therefore, the orientation of all the particles were the same, we should, if many particles were sending light up the tube, get no black spot in the focal plane, because the black spot, supposing there to be one, due to one particle, would not coincide with that due to another. And further, even if the orientation of all the particles were the same, and if every particle alone did send off no light in some particular direction, so that there were a black spot in the second focal plane, then, unless the common orientation were such that, for every plane of polarisation of the incident light, the black spot were in the same plane as if the particles were spheres, which is an impossibility, spheroidal or crystalline particles could not account for the effect observed.

These considerations show, therefore, that the small particles in gold ruby glass are really spheres of gold, so long as their dimensions are considerably smaller than $0{\cdot}1\mu$ (10^{-5} centim.).

This result is of considerable interest in connection with the formation of crystals. When a metal crystallises out of a vitreous solution, it appears that until the dimensions have increased beyond a certain limit, the forces of surface tension overcome the crystallic forces, and the particles of metal are spherical and not crystalline.*

Mr. G. T. BEILBY has arrived at the same conclusion from microscopic examination of the films of metal deposited from solutions ('Proc. Roy. Soc.,' vol. 72, 1903, p. 223).

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* [*Note added 14th May*, 1904.—The presence of crystals, whether of silicates or of reduced metal, in many pottery glazes suggests that minute spheres of the same material as the crystals were present before the formation of these crystals, and that some may co-exist with the crystals. The colours of the glazes may therefore be wholly or in part due to the presence of these minute spheres, in the same manner as a gold ruby glass depends for its colour on the presence of minute spheres of gold.]

In the manufacture of gold and copper ruby glasses and of silver glass, the gold or copper or silver is mixed with the other ingredients of the glass before the first firing. If, when the glass is formed in the furnace, the whole be quickly cooled, the glass with the metal in it is colourless and exactly resembles clear glass. I have had in my possession several pieces of such clear gold glass, and some of clear silver glass. One of the former was used in an experiment with the emanation from radium, to be described later.

In this clear glass the gold or silver is probably in solution in the glass. But when the glass is re-heated the metal "crystallises" out of solution, or, as we shall say, is "excreted" from the glass and appears in the small particles observed by SIEDENTOPF and ZSIGMONDY. These particles of metal, as we shall show, account for the colour of the glass.

I have seen a piece of copper glass which was allowed to cool down slowly in the glass pot along with the furnace, taking a week or more in the process. The glass formed a dark brown, nearly opaque, mass with minute crystals of bright, shining copper scattered throughout its substance, the crystals being large enough to be easily distinguishable with the naked eye, while the appearance of the whole mass somewhat resembled that of the well-known African stone, aventurine.

It is suggested that the second heating, without melting the glass, confers sufficient freedom on the molecules of the glass to enable the forces of surface tension to exert themselves in bringing the molecules of the metal, which have been distributed amongst those of the glass, together into heaps, the phenomenon being similar to that exhibited when a metal film is heated to 300° or 400° without being melted, when, as will be described later, the metal forms itself into minute granules, which, in the light of what we have proved for the particles in gold glass, must be spheres or spheroids with axes normal to the film. The latter form is possible for the films of metal, though not for the metal in the gold glass, because a thin film, as opposed to a piece of glass, is not subjected to similar conditions in all directions.

§ 4. We have thus to consider the problem of light traversing a medium containing many small metal spheres to a wave-length of light.

It has been seen (§ 2) that a small metal sphere produces in all surrounding space the same effect as would be produced by a Hertzian doublet placed at its centre. We may therefore imagine the spheres replaced by such electric doublets and thus avoid considering their finite size.

Let the average (for a large number of doublets) moment of a doublet be, at time t, $\mathbf{f}(t) \equiv \{f_1(t), f_2(t), f_3(t)\}$.

Then if there be $\mathfrak{N}$ spheres per unit volume, the polarisation of the medium will be $\mathbf{f}'(t) = \mathfrak{N}\mathbf{f}(t)$. If $\mathbf{E}'$, due to the regular force $\mathbf{E}_0$ together with forces due to the neighbouring doublets, be the force causing the polarisation $\mathrm{f}(t)$, then we have proved (§ 2) that

$$\mathrm{f}(t) = a^3 \frac{N^2 - 1}{N^2 + 2} E'.$$

Now by means of the analysis given by H. A. LORENTZ ('Wied. Ann.,' 9, 1879, p. 641) and by LARMOR ('Phil. Trans.,' A, 1897, p. 238), and which has been fully verified in LORENTZ's own paper and by others, it can be proved that (see § 7 below)

$$E' = E_0 + \frac{4\pi \mathrm{f}'}{3} = E_0 + \frac{4\pi}{3} \mathfrak{N} a^3 \frac{N^2 - 1}{N^2 + 2} E' \quad . \quad . \quad . \quad . \quad . \quad (9),$$

provided the medium under consideration extends throughout a space of dimensions which in no direction are of an order of magnitude so small as a wave-length of light. This provision is satisfied except in the case of very thin films. When dealing with such films in a later portion (§ 7) of this paper we shall return to the consideration of this point.

From equation (9) we obtain

$$\mathrm{E}' = \mathrm{E}_0 \Big/ \left\{ 1 - \frac{4\pi}{3} \mathfrak{N} a^3 \frac{N^2 - 1}{N^2 + 2} \right\},$$

so that

$$\mathrm{f} = \mathrm{E}_0 \frac{N^2 - 1}{N^2 + 2} a^3 \Big/ \left\{ 1 - \frac{4\pi}{3} \mathfrak{N} a^3 \frac{N^2 - 1}{N^2 + 2} \right\}.$$

CLERK MAXWELL's equations written with Hertzian units for this medium, now, therefore, are

$$\epsilon' \frac{d\mathrm{E}}{dt} = c \operatorname{curl} \mathrm{H} \quad \text{and} \quad \frac{d\mathrm{H}}{dt} = - c \operatorname{curl} \mathrm{E},$$

where

$$\epsilon' = (\mathrm{E} + 4\pi \mathrm{f}')/\mathrm{E} = 1 + 4\pi \mathfrak{N} a^3 \frac{N^2 - 1}{N^2 + 2} \Big/ \left\{ 1 - \frac{4\pi}{3} \mathfrak{N} a^3 \frac{N^2 - 1}{N^2 + 2} \right\}.$$

We have therefore proved that a medium consisting of small metal spheres distributed *in vacuo*, many to a wave-length of light, is optically equivalent to a medium of refractive index n' and absorption κ' given by $N' \equiv n'(1 - \iota\kappa') \equiv \sqrt{\epsilon'}$, where

$$\epsilon' = 1 + \frac{4\pi \mathfrak{N} a^3 \dfrac{N^2 - 1}{N^2 + 2}}{1 - \dfrac{4\pi}{3} \mathfrak{N} a^3 \dfrac{N^2 - 1}{N^2 + 2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

We shall throughout use the symbol μ to denote the volume of metal per unit volume of the medium (except when μ is evidently used to denote the thousandth part of a millimetre). Thus $\mu \equiv \frac{4\pi}{3}\mathfrak{N}a^3$, and equation (10) becomes

$$\epsilon' = 1 + \frac{3\mu\dfrac{N^2-1}{N^2+2}}{1-\mu\dfrac{N^2-1}{N^2+2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10').$$

If the metal spheres be situated in glass of refractive index ν instead of *in vacuo*, this equation becomes

$$\{n'(1-\iota\kappa')\}^2 = \epsilon' = \nu^2 + \frac{3\mu\nu^2\dfrac{N^2-\nu^2}{N^2+2\nu^2}}{1-\mu\dfrac{N^2-\nu^2}{N^2+2\nu^2}} \quad . \quad . \quad . \quad . \quad (11).^*$$

The constants n' and κ' of the medium thus depend only on μ, the relative volume of metal, and not on the radii of the individual spheres. It is clear that the spheres may now be supposed to be of quite various radii, provided only that there be many spheres to a wave-length of light in the medium.

We have given the general result which holds for all values of μ, as we shall require it later. But in the case of metal glasses, by which name we shall describe glasses in which a metal is present in metallic form, the value of μ varies from about 10^{-4} for a silver glass down to about 10^{-6} for a soda glass coloured by radium. The last equation giving the optical constant $N' = n'(1-\iota\kappa')$ of the metal glass may be written

$$\{n'(1-\iota\kappa')\}^2 - \nu^2 = 3\mu\nu^2\frac{N^2-\nu^2}{N^2+2\nu^2} \equiv 3\mu\nu^2(\alpha - 2\beta\iota), \text{ say} \quad . \quad . \quad (12),$$

where N is the optical constant of the metal and ν the index of refraction of the glass by itself.

§ 5. Equation (12) may now be written

$$n'^2(1-\kappa'^2) - \nu^2 - 2\iota n'^2\kappa'$$
$$= 3\,\mu\nu^2\frac{n^2(\kappa^2-1)+\nu^2+2\iota n^2\kappa}{n^2(\kappa^2-1)-2\nu^2+2\iota n^2\kappa} \equiv 3\mu\nu^2(\alpha - 2\iota\beta).$$

Thus, equating real and imaginary parts, we find, after some reduction,

$$\left.\begin{aligned} \alpha &\equiv \frac{n'^2(1-\kappa'^2)-\nu^2}{3\mu\nu^2} = \frac{\{n^2(\kappa^2-1)\}^2 - n^2(\kappa^2-1)\nu^2 + 4n^4\kappa^2 - 2\nu^4}{\{n^2(\kappa^2-1)-2\nu^2\}^2 + 4n^4\kappa^2} \\ \beta &\equiv \frac{n'^2\kappa'}{3\mu\nu^2} = \frac{3n^2\kappa\nu^2}{\{n^2(\kappa^2-1)-2\nu^2\}^2+4n^4\kappa^2} \end{aligned}\right\} . \quad (13).$$

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* [*Note added* 16*th May*, 1904.—This equation may be written $\frac{N'^2-\nu^2}{N'^2+2\nu^2} = \mu\frac{N^2-\nu^2}{N^2+2\nu^2}$].

We have now to see whether by means of these equations (13), and of the values of n and κ for various metals, we shall be able to predict the colour of a glass which contains a number of small metal spheres, whose linear dimensions and distances apart are small compared with a wave-length of visible light.

In the annexed table the refractive index of the glasses has been taken to be $\nu = 1{\cdot}56$.

The values of $n^2(\kappa^2 - 1)$ and of $n^2\kappa$ for the metal are those given by DRUDE ('Physikalische Zeitschrift,' January, 1900), for yellow light ($\lambda = {\cdot}0000589$ centim.), and for red light ($\lambda = {\cdot}0000630$ centim.). For the potassium-sodium amalgam, however, blue and yellow light were used instead of yellow and red.

Now let us suppose that μ, the quantity of metal per unit volume, is very small If, then, α and β represent the numbers in the penultimate and last columns of Table I. respectively, we have

$$n'^2(1 - \kappa'^2) = \nu^2 + 3\mu\nu^2\alpha, \qquad n'^2\kappa' = 3\mu\nu^2\beta.$$

Hence

$$n'^2(1 + \kappa'^2) = \{\nu^4 + 2\nu^2 . 3\mu\nu^2\alpha + 9\mu^2\nu^4\alpha^2 + 36\mu^2\nu^4\beta^2\}^{\frac{1}{2}}$$

$$= \nu^2 + 3\mu\nu^2\alpha + 9\mu^2\nu^2 2\beta^2 + 27\mu^3(\ldots) + \ldots$$

Hence, neglecting higher powers of μ,

$$n'\kappa' = 3\mu\nu\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Now, suppose that light of wave-length λ *in vacuo* travels through this composite medium, whose constants are n' and κ'. The light *in vacuo* being given by

$$X = A \exp \{2\pi\iota(t/T - z/\lambda)\},$$

in this medium it is given by

$$X = A' \exp \{2\pi\iota(t/T - N'z/\lambda)\}$$

$$= Ae^{-\frac{2\pi n'\kappa' z}{\lambda}} \exp \{2\pi\iota(t/T - n'z/\lambda)\},$$

so that $n'\kappa'$ measures the absorption. In fact, the intensity of the light sinks to e^{-2} (= $\frac{1}{7{\cdot}4}$ nearly) of its original value in traversing a distance

$$d \equiv \frac{\lambda}{2\pi n'\kappa'} = \frac{\lambda}{6\pi\mu\nu\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

of the medium.

We have now to apply the formulæ to the observations, in order to test the validity of our analysis as regards the actual phenomena. SIEDENTOPF and ZSIGMONDY give ('Ann. der Physik,' January, 1903, pp. 33, 34) a table of various gold glasses examined by them. This table is reproduced in Table II.

TABLE I

λ yellow = ·0000589 centim. λ red = ·0000630 centim.

Metal.	Coloured	$\{n^2(\kappa^2-1)\}^2$.	$n^2(\kappa^2-1)\nu^2$.	$4n^4\kappa^2$.	Numerator of first fraction in (2).	$3n^2\kappa\nu^2$.	$\{n^2(\kappa^2-1)-2\nu^2\}^2$.	Denominator.	$\frac{n'^2(1-\kappa'^2)-\nu^2}{3\mu\nu^2}\equiv\alpha$	$\frac{n'^2\kappa'}{3\mu\nu^2}\equiv\beta$.
Silver*	Yellow	182·25	32·8536	1·7956	139·3472	4·8915	74·5287	76·3243	1·826	·0641
	Red	246·49	38·2075	2·6244	199·0631	5·9136	78·0219	80·6463	2·468	·0733
Copper	Yellow	42·25	15·8184	11·56	26·1468	12·4114	2·6667	14·2267	1·838	·8724
	Red	90·25	23·1192	12·99	58·2760	13·1414	21·4647	34·4547	1·691	·3814
Gold*	Yellow	60·84	18·9821	4·2436	34·2567	7·5198	8·6025	12·8461	2·667	·5854
	Red	96·04	23·8493	3·6864	64·0323	7·0088	24·3345	28·0209	2·285	·2501
Potassium-sodium.	Blue	9·61	7·5442	·2704	− 9·5086	1·8982	3·1223	3·3927	−2·803	·5594
	Yellow	22·09	11·4379	·2916	− ·9011	1·9712	·0286	·3196	−2·819	6·1678

$\nu = 1·56.$ $\nu^2 = 2·4336.$ $2\nu^2 = 4·8672.$ $2\nu^4 = 11·8448.$

* See also Appendix.

TABLE II.

Glass.	Colour by transmitted light. (Layer about 4 millims. thick).	Colour of cone of light.	Behaviour of cone of light when examined with a Nicol.		Total gold content, μ in cub. millims.; gold per cub. millim., glass.	Colorimetrically found gold content, μ.	Size of particles in $\mu\mu$.	
			Nicol parallel to plane of incidence.	Nicol perpendicular to plane of incidence.			(*a*) From gold content.	(*b*) From colorimetrical gold content.
A*	Colourless . . .	Gold-yellow . .	White-yellow	Red - yellow, rather lessened	12·6.10^{-6}	—	487–791	—
B*	Dirty reddish, "lebrig" cloudy	Gold - yellow, very intense	White-yellow	Red - yellow, rather lessened	10·1.10^{-6}	—	131–173	—
Ca*	Almost colourless	Green, intense .	Green, scarcely lessened	Green, much lessened .	13·3.10^{-6}	—	115–145	—
Cb*	Pink.	Green	Green, scarcely lessened	Green, much lessened .	13·3.10^{-6}	—	63–106	—
Cc*	Pink.	Green	Green	Green, almost extinguished	13·3.10^{-6}	1·34.10^{-6}†	20·6–32·8	9·6–15·3
Da	Clear blue . . .	Pink, copper-red	As without Nicol . .	Brown, lessened . . .	6·8.10^{-6}	—	68·8–103	—
Db	Clear blue to violet	Brass-yellow . .	As without Nicol . .	Brown-red, lessened .	6·8.10^{-6}	—	68–74	—
E	Blue-violet with pink films	Brown with green films	As without Nicol, the green films rather brighter than the brown	Brown, lessened; green extinguished	9·05.10^{-6}	4·5.10^{-6}	13·2–17·4	10·5–13·9
F	Clear red . . .	Green, slight intensity	As without Nicol . .	Almost extinguished .	8·0.10^{-6}	4·4.10^{-6}	9·3–12·5	7·6–10·2
G	Deep red, intense	Green, feeble intensity	As without Nicol . .	Extinguished in places.	14·0.10^{-6}	7·2.10^{-6}	10·5–13·2	8·4–10·6
H	Pink.	Green, slight intensity	As without Nicol . .	Extinguished in places.	—	1·0.10^{-6}	—	3·9–6·9‡

* Distances of particles apart fully resolvable.
† This value is probably too small.
‡ Distribution was very uniform in this glass.

If l^3 denote the volume of the gold particles, so that l is given in the 8th or 9th column of the Table II., according as μ, the gold content, is taken from the 6th or 7th column, then the number $\mathfrak{N}$ of gold particles per cubic (10^{-4} centim.) is given by $\mathfrak{N} = 10^9 \mu / l^3$.

We have said that for our analysis to be applicable there must be many spheres to a wave-length. Since the spheres in glasses A, B, Ca, Cb, Cc (Table II.) can be separated by a Zeiss $\frac{1}{12}$th objective, they must be at a distance apart greater than $\cdot 2\mu$, or half a wave-length of violet light. We shall therefore not expect our analysis to apply to them. It is further apparent that the particles are more widely separated in Da than Db. If we take for l the mean of the two numbers given, we have

$$\mathfrak{N}_{\text{Db}} = \cdot 019, \quad \mathfrak{N}_{\text{E}} = 2\cdot 48, \quad \mathfrak{N}_{\text{F}} = 6\cdot 24, \quad \mathfrak{N}_{\text{G}} = 8\cdot 40, \quad \mathfrak{N}_{\text{H}} = 6\cdot 35,$$

so that, since $\mathfrak{N}_{\text{F}}$, $\mathfrak{N}_{\text{G}}$, and $\mathfrak{N}_{\text{H}}$ are larger than $\mathfrak{N}_{\text{Db}}$ and $\mathfrak{N}_{\text{E}}$, the glasses F, G, H satisfy our condition best.

But here we are presented with a difficulty. The wave-length of the yellow light, $\lambda = \cdot 0000589$ centim., is in our glass ($\nu = 1\cdot 56$) only, $\lambda' = \cdot 00003775$ centim. or $\cdot 3775\mu$. Thus to find the number of gold particles in λ'^3, we must multiply $\mathfrak{N}$ by $(\cdot 3775)^3 = \cdot 0538$. We shall thus, even for glass G, have less than one particle to a wave-length. On the other hand, SIEDENTOPF warns us (*loc. cit.*, p. 27) that the linear dimensions of the particles are only to be taken as upper limits and may be three times too large.

Suppose this is the case, then the number of particles in a yellow wave-length in the glass is $27 \times \cdot 0538$ $(= 1\cdot 45)$ times the above numbers, $\mathfrak{N}$, with of course a still greater value for red light.

On this hypothesis then the glasses F, G, H alone of the series satisfy our condition. If, therefore, the theory is correct, it should explain the colour and other optical properties of these three glasses as set out in the first five columns of Table II.

Let us, for instance, consider the colour of glass G. From equation (15) we have as the distance d in which the intensity of light of wave-length λ is reduced to $1/7\cdot 4$ of its original value $\lambda/6\pi\mu\nu\beta$ given by (15).

From the Table I. we have, supposing $\nu = 1\cdot 56$,

$$\beta = \cdot 5854, \quad \text{for yellow light} \quad \lambda = 10^{-7}\ \cdot 589 \text{ centim.}$$
$$\beta = \cdot 2501, \quad \text{for red light} \quad \lambda = 10^{-7}\ \cdot 630 \quad ,,$$

The value of μ found by colorimetry is 72.10^{-7}. Consequently we have

Yellow, $d = \ \cdot 48$ centim., nearly.
Red, $d = 1\cdot 19$ centims. ,,

Since the latter number is greater than the former, it follows that this glass, and indeed all gold glasses for which our condition is satisfied, should be much more red than yellow. Presumably, therefore, they are still more yellow than green, and more green than blue.* We should therefore expect the gold glasses F. G, H. which satisfy our condition, to be *red*, as in fact they are.

The above values for d are certainly of the right order, but they may be somewhat too large. If we had taken the value of μ, called the Total Gold Content in Table II., the corresponding values of d would have been only half those given above.

It is to be remembered that manufacturers, in making gold ruby glass for "flashing" on to clear glass, use much more gold. A common value for the total gold content is about 3.10^{-5}.

By means of equation (15) and Table I. we can in this way predict whether a gold. silver, or copper glass for which $\nu = 1{\cdot}56$ will transmit more red or more yellow light, and whether such a glass containing small spheres of "potassium-sodium" will transmit more yellow or more blue.

We thus find that when there are several metal spheres to a wave-length

Silver glass transmits yellow (β/λ yellow $<$ β/λ red),
Copper ,, ,, red (β/λ red $<$ β/λ yellow),
Gold ,, ,, ,, (β/λ ,, $<$ β/λ ,,),
Potassium-sodium glass transmits blue (β/λ blue $<$ β/λ yellow).

From the values of β on Table I., p. 396, we see that for a silver glass to absorb as much red light as a gold glass does yellow, μ would have to be $\frac{5854}{733}$, or, roughly, eight times as great for the silver glass as for the gold. And, since the values of β for yellow and red light are more nearly equal for silver than for gold, in order to produce the same coloration there would have to be even more than eight times as much silver (by volume) as gold. I am told that manufacturers put in ten times as much silver by weight into a silver glass as they put gold into a gold glass.

Again, the very large value of β for yellow light in a potassium-sodium glass shows that such a glass would absorb as much yellow light as a gold glass with only $\frac{1}{10}$ of the amount of metal excreted.

Thus a very slight excretion of the potassium-sodium metal would give a very strong blue or violet coloration. This probably explains the colouring of soda glass by radium, the radiation causing the excretion of the metal.

In order to test this hypothesis I asked Mr. F. SODDY, on 9th November, 1903, at University College, London, to examine whether the emanation from radium was capable of colouring quartz glass in which there could evidently be no possibility of the excretion of metal. He stated that he and Professor RAMSAY had already made this experiment and had found no coloration.

At my request Mr. SODDY then placed a small piece of colourless gold glass in a tube containing some emanation. Within two days an unmistakable ruby tint appeared in the glass.*

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* See Appendix.

It seems probable that the violet coloration of soda glass bulbs used in the production of Röntgen rays may be due to the excretion of metal caused by the β rays from the cathode.

The observations of ELSTER and GEITEL, 'Wied. Ann.,' 59, p. 487, 1896, quoted by J. J. THOMSON, 'Conduction of Electricity through Gases,' p. 496, that salts of the alkali metals coloured by exposure to cathode rays exhibit photo-electric effects, suggestive of the presence of traces of the free metal, support this view as to the cause of the coloration of metal glasses exposed to the radiation from radium.

From equation (6), as modified for the case when the metal sphere is surrounded by glass of refractive index ν, it appears that the amplitude at any point of the light emitted from the sphere is proportional to $\left|\frac{N^2 - \nu^2}{N^2 + 2\nu^2}\right|\frac{a^3}{\lambda^2}$, where $|u + \iota v|$ denotes the modulus, $+\sqrt{u^2 + v^2}$. Using α and β as defined in equation (13), we have $\left|\frac{N^2 - \nu^2}{N^2 + 2\nu^2}\right|^2 \equiv \alpha^2 + 4\beta^2$, where α and β are to be found from the table on p. 396, where $\nu = 1{\cdot}56$. Thus at any point the intensity of light emitted by a sphere of radius a is proportional to $(\alpha^2 + 4\beta^2)/\lambda^4 \equiv \mathrm{I}$, say. Measuring λ in millim./1000, the Table I. gives the following values of I :—

		Silver.	Copper.	Gold.
Yellow ($\lambda = {\cdot}589$) I_y	=	27·95	62·11	70·88
Red ($\lambda = {\cdot}630$) I_r	=	38·81	21·75	34·79.

From these values of I it appears that when white light falls on a small sphere the light emitted is, for

Silver, more red than yellow, $\mathrm{I}_r > \mathrm{I}_y$,
Copper ,, yellow ,, red, $\mathrm{I}_y > \mathrm{I}_r$,
Gold ,, ,, ,, ,, $\mathrm{I}_y > \mathrm{I}_r$.

The presumption is that for the two latter the light may be more green than yellow.

In the table given by SIEDENTOPF and ZSIGMONDY (*loc. cit.*), of which a copy is given (Table II., p. 397), it is seen that of the five glasses Cc, E, F, G, H, whose particles are small compared with a wave-length of light in the glass, the four glasses Cc, F, G, H contain particles which send out a *green* cone of light, and the glass E contains some particles which send out *green* and some which send out *brown*.

Thus far we have confined attention to glasses for which the condition of having many metal particles to a wave-length is satisfied. We have shown that when the metal is gold such glasses should be pink (*cf.* column 3 of Table II.) by transmitted light; and that the small gold spheres should send up the microscope light which is pre-eminently yellow or green (*cf.* columns 4 and 5); and we have remarked that for the same reason that explains the polarisation of sky light, such small spheres send no light directly up the microscope tube when the electric vector of the incident light is in that direction, so that in this case the cone of light as examined with the low-power objective will be cut off (*cf.* column 5), although the large numerical aperture of the Zeiss $\frac{1}{12}$th oil immersion lens will allow some light to go up the tube, but so as to leave a black spot in the centre of the focal plane of the microscope as shown in fig. 6.

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* [*Note added* 14*th May*, 1904.—Sir WILLIAM RAMSAY has lately exposed some clear silver glass and some soda glass at the same time to the emanation from radium. After a fortnight's exposure the silver glass had turned a faint yellow and the soda glass a deep blue-violet.]

All these deductions from our analysis are confirmed in every detail by the three glasses F, G, H (Table II.). And it is these very glasses, of all the glasses in that table, for which, according to the numbers there given, the particles are both smallest and closest together.

§ 6. Let us now briefly notice the remaining glasses of Table II. For these glasses the number of metal particles to a wave-length, measured by (gold content)$^{\frac{1}{3}}$ ÷ size of particle, as determined from the 6th and 8th or from the 7th and 9th columns of that table, is smaller than for the glasses F, G, H, which show the regular pink colour. For the glasses A to E this number is greatest for the glasses Cc and E, of which the former and parts of the latter *do* show the regular pink colour.

Even glasses which do not satisfy the condition of many particles to a wave-length, and which consequently do not exhibit the "regular" (pink) colour of gold glass, have many of their properties co-ordinated by the results we have obtained for regular glasses.

Take, for instance, the glasses A and B (Table II.). Comparison of the gold content μ with the size of the *observed* particles shows that those particles at any rate are so far apart as not to satisfy our condition. The fact that glass A is colourless shows that if there are also minute spheres present which escaped observation, they also lie so far apart as not to be many to a wave-length. On the other hand the pink colour of glass B suggests the presence of minute unobserved spheres which are sufficiently close together to satisfy our condition, the absorption of the glass being proportional to that small part of the gold content (μ) which is associated with the minute spheres.

In both glasses the large particles reflect much more light than is emitted by the minute spheres. The colour of this reflected light is the usual yellow-red metallic reflection from gold. Therefore the colour of the cone of light should be gold-yellow (i).

When the Nicol is introduced parallel to the plane of incidence, presumably half the incident light is cut off. Consequently the large particles send only half the yellow-red light up the tube that they previously sent. Owing, however, to the fact that the minute spheres send no light directly up the tube when the electric vector of the incident light is parallel to the microscope tube (Nicol perpendicular to plane of incidence), less than half the green light from any small spheres will be cut off. The cone of light will therefore have more green in proportion to the yellow-red than before the introduction of the Nicol. Therefore the colour of the cone of light will be more white than before (ii).

When the Nicol is perpendicular to the plane of incidence, the green light from the small spheres is cut off, so the colour of the cone of light will be more red than with no Nicol, and therefore the total quantity of light sent up the tube will be rather lessened (iii). The conclusions (i), (ii), (iii) are in accordance with the phenomena tabulated in the 3rd, 4th, and 5th columns of Table II.

The glasses Ca, Cb, Cc present no special difficulties. We have seen (§ 3) that those metal particles in a gold glass whose diameters are less than $0{\cdot}1\mu$ (10^{-5} centim.) are spherical, and (§ 5) that small gold spheres send *green* light up the microscope tube. In the above-named glasses the figures in the 7th column of Table II. show that the particles are so small as to approximate to the spherical form. This is confirmed by the green cone of light and its approximate extinction when the electric vector of the incident light is in the direction of the microscope tube.

As here, too, the observed particles are far enough apart to be distinguished under the microscope, it is necessary to postulate additional minute spheres to explain the pink colours of these glasses.

In glasses D and E the blue and violet colours of the transmitted light present a difficulty which I have not yet been enabled completely to surmount.* It is probable that the particles in this glass are not sufficiently thickly distributed to satisfy the condition of there being many particles to a wave-length of blue light. When the incident light is blue, the absorption that we have investigated is therefore not present. When, however, the incident light is red, there are sufficient particles to a wave-length for absorption to take place. Thus, although if light of all wave-lengths were absorbed, the red would be least absorbed; yet here it is only the larger wave-lengths that suffer the absorption whose nature we have investigated.

PART II.

§ 7. With a view to examining whether these principles apply to the colour changes exhibited by translucent films of metal when heated, observed by Mr. G. T. BEILBY ('Roy. Soc. Proc.,' vol. 72, 1903, p. 226) and by Professor R. W. WOOD ('Phil. Mag.,' vol. 3, 1902, p. 396), we proceed to consider the transmission of light by films of metal, the metal being in the form of small spheres, many to a wave-length of light in the film

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* See Appendix added July 8th for explanation of Blue and Violet Colours.

We shall first confine attention to very thin films, defining very thin films to be such that $\pi d/\lambda'$ may be treated as small, d being the thickness of the film, and λ' the wave-length of light in the film.

It has already been noticed that equation (9), p. 393, does not hold for very thin films. That equation is obtained by observing that the *average* action of its neighbours on a particle is that due to a medium which is perfectly uniformly polarised in the neighbourhood of the particle, and whose external boundary is that of the actual medium, and whose internal boundary is a sphere of radius r_0, equal to the smallest distance between the centres of two particles. POISSON has shown that the effect of such a uniformly polarised medium is equivalent to that of a surface distribution over its internal and external boundaries.

The medium actually present here can only be treated as uniformly polarised throughout the region inside a sphere whose radius, r_1, is small compared with the wave-length of light in the medium. When the outer boundary of the medium is in all directions many wave-lengths distant from the particle under consideration, the effect of the periodically varying polarisation outside $r = r_1$ can be allowed for by neglecting the Poisson distribution on the outer boundary of the medium. Consequently, in this case, the effect on any particle of the remaining particles is that due to a Poisson distribution over the sphere $r = r_0$, which leads to equation (9).

When the external boundary of the medium is, in any direction, at a very small distance from the average particle, we are not justified in neglecting the Poisson distribution over that boundary. In the case of a thin film of the medium in the plane of xy it is, however, clear that when the electric force is parallel to that plane, there is no Poisson distribution over the surfaces of the film. Consequently the film has (complex) dielectric constants in the direction of the axes of x and y, which are the same as for the medium in bulk. Omitting the accent in equation (10′), this constant is given by

$$\epsilon - 1 = 3\mu \frac{\mathrm{N}^2 - 1}{\mathrm{N}^2 + 2} \bigg/ \left\{1 - \mu \frac{\mathrm{N}^2 - 1}{\mathrm{N}^2 + 2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

The dielectric constant ϵ', parallel to $0z$, may be different from ϵ; if so, the film behaves optically like a uniaxial crystal whose three (complex) dielectric constants are ϵ, ϵ, ϵ', the optic axis being normal to the film.

§ 8. Putting $\nu = 1$ in equations (12) and (13), we have

$$\frac{\mathrm{N}^2 - 1}{\mathrm{N}^2 + 2} = \alpha - 2\beta\iota \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12'),$$

where

$$\alpha = \frac{\{n^2(\kappa^2 - 1)\}^2 - n^2(\kappa^2 - 1) + 4n^4\kappa^2 - 2}{\{n^2(\kappa^2 - 1) - 2\}^2 + 4n^4\kappa^2}, \quad \beta = \frac{3n^2\kappa}{\{n^2(\kappa^2 - 1) - 2\}^2 + 4n^4\kappa^2} \cdot \quad (13').$$

We shall henceforward find it convenient to use n and κ to denote the constants of the medium containing the spheres. The constants of the metal itself will therefore be denoted by n_1 and κ_1, and it will appear, as is *à priori* evident, that the latter are the values of n and κ when $\mu = 1$. Since therefore $\epsilon \equiv \{n(1 - \iota\kappa)\}^2$, equation (16) gives us on substituting from (12′)

$$n^2(1-\kappa^2) - 1 - 2\iota n^2\kappa = \epsilon - 1 = 3\,\frac{\mu\alpha - 2\mu\beta\iota}{1 - \mu\alpha + 2\mu\beta\iota} \quad . \quad . \quad . \quad (17),$$

from which, by equating real and imaginary parts,

$$n^2\kappa = \frac{3\mu\beta}{(1-\mu\alpha)^2 + 4\mu^2\beta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

$$n^2(\kappa^2 - 1) = 2 - \frac{3(1-\mu\alpha)}{(1-\mu\alpha)^2 + 4\mu^2\beta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

whence

$$\{n^2(\kappa^2+1)\}^2 = 3\,\frac{3 - 4(6 - \mu\alpha)}{(1-\mu\alpha)^2 + 4\mu^2\beta^2} + 4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

The following table gives the values of α and β as found by means of formula (13′) from the constants n and κ of the solid metal as given by DRUDE (*loc. cit.*) :—

TABLE III.

Metal.	Colour.	α.	β.
Gold	Yellow $\lambda = \cdot589$	1·4593	·0816
	Red $\lambda = \cdot630$	1·3626	·0446
Silver	Yellow $\lambda = \cdot589$	1·2574	·0150
	Red $\lambda = \cdot630$	1·2160	·01277
Potassium-sodium	Blue	3·269	·531
	Yellow	2·068	·107

In order to determine the values of n and κ for various values of μ, the numerical values of the functions ξ, η, ζ, where

$$\frac{\xi}{\mu\beta} = \frac{\eta}{1-\mu\alpha} = \frac{\zeta}{1} = \frac{1}{(1-\mu\alpha)^2 + 4\mu^2\beta^2} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

were calculated for gold and for silver for the following values of μ:

$$\mu = \cdot 1, \quad \mu = \cdot 5, \quad \mu = \cdot 6, \quad \mu = \cdot 7, \quad \mu = \cdot 8, \quad \mu = \cdot 9, \quad \mu = 1 \cdot 0.$$

Equations (19) and (20) may be written

$$n^2(\kappa^2 - 1) = 2 - 3\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19'),$$

$$n^2(\kappa^2 + 1) = \{3(3\zeta - 4\eta) + 4\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20'),$$

whence the values of n and $n\kappa$ for gold and for silver were calculated for the above values of μ. The values of $n^2\kappa$ thence obtained were checked against those obtained by means of equation (18), namely,

$$n^2\kappa = 3\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18').$$

In the case of silver with μ less than $\cdot 8$ it was, however, seen to be better to obtain $n\kappa$ as the quotient of the value of $n^2\kappa$ got from (18′), by the value of n got from (19′) and (20′), owing to the large probable error when $n\kappa$ was determined directly from (19′) and (20′).

From equation (13′) we find

$$n_1^2\kappa_1 = \frac{3\beta}{(1-\alpha)^2 + 4\beta^2}, \quad n_1^2(\kappa_1^2 - 1) = 2 - \frac{3(1-\alpha)}{(1-\alpha)^2 + 4\beta^2},$$

which are the same as equations (19) and (20) with $\mu = 1$. Consequently, as should be the case, the medium of spheres is equivalent to the solid metal wherein the spheres are of such varied sizes that they fill the whole space. Another check on the tabulated numbers is afforded therefore by a comparison of the calculated and observed values of $n_1^2\kappa_1$, $n_1\kappa_1$ and n_1.

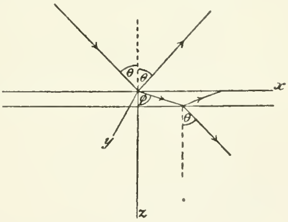


Fig. 10.

I believe that nearly all the numbers here given for silver and for gold are subject to an error of less than 1 per cent.

The values of $n^2\kappa = 3\xi$ and of η for the potassium-sodium amalgam of DRUDE's table, 'Phys. Zeitschrift,' January, 1900, are less carefully calculated.

§ 9. Consider now the incidence of plane polarised light on a plate of this medium. We shall first suppose the plate to be very thin and therefore optically crystalline.

Suppose the two surfaces of the film are $z = 0$ and $z = d$, and that zx is the plane of incidence.

TABLE IV.—Gold and Silver.

		Colour of incident light.	$\mu = \cdot 1$.	$\mu = \cdot 5$.	$\mu = \cdot 6$.	$\mu = \cdot 7$.	$\mu = \cdot 8$.	$\mu = \cdot 9$.	$\mu = 1\cdot 0$.	Given value for solid metal.	Value for $\mu = \frac{1}{\alpha}$.
Gold	$n\kappa$	Yellow, $\lambda = \cdot 589$	·027	·528	1·50	4·07	3·80	3·19	2·82	2·82	3·80 for $\mu = \cdot 685$
		Red, $\lambda = \cdot 630$	·014	·240	·605	2·87	5·01	3·75	3·15	3·15	4·89 for $\mu = \cdot 734$
	n	Yellow	1·23	2·91	3·89	3·12	1·14	·58	·37	·37	3·53 for $\mu = \cdot 685$
		Red	1·21	2·70	3·67	5·40	1·62	·57	·31	·31	4·68 for $\mu = \cdot 734$
	$n^2\kappa$	Yellow	·034	1·53	5·86	12·70	4·34	1·84	1·03	1·03	13·42 for $\mu = \cdot 685$
		Red	·018	·65	2·22	15·53	8·11	2·09	·96	·96	22·9 for $\mu = \cdot 734$
	M_0	Yellow	·979	·578	·406	·374	·695	1·05	1·34	—	·354 for $\mu = \cdot 685$
		Red	·982	·622	·450	·247	·435	·845	1·19	—	·232 for $\mu = \cdot 734$
Silver	$n\kappa$	Yellow	·0049	·066	·140	·449	8·75	4·91	3·68	3·67	7·99 for $\mu = \cdot 795$
		Red	·0042	·052	·104	·282	3·00	5·69	3·97	3·96	8·51 for $\mu = \cdot 822$
	n	Yellow	1·20	2·46	3·19	4·74	6·74	·46	·18	·18	7·87 for $\mu = \cdot 795$
		Red	1·19	2·38	3·01	4·24	8·81	·64	·20	·20	8·39 for $\mu = \cdot 822$
	$n^2\kappa$	Yellow	·0059	·163	·445	2·13	58·9	2·24	·67	·67	62·9 for $\mu = \cdot 795$
		Red	·0050	·124	·313	1·19	26·5	3·65	·81	·81	71·4 for $\mu = \cdot 822$

Potassium-Sodium.

	Colour of incident light.	$\mu = \cdot 1$.	$\mu = \cdot 2$.	$\mu = \cdot 4$.	$\mu = \cdot 5$.	$\mu = \cdot 6$.	$\mu = \cdot 7$.	$\mu = \cdot 8$.	$\mu = \cdot 9$.	$\mu = 1\cdot 0$.	Given value for solid metal.	Value for $\mu = \frac{1}{\alpha}$.
$n^2\kappa$	Blue	·343	1·29	2·32	1·17	·717	·512	·403	·309	·254	·26	4·85, $\mu = \cdot 306$
	Yellow	·051	—	3·45	12·38	3·26	1·10	·59	·48	·270	·27	14·49, $\mu = \cdot 484$

First let the incident light be polarised in the plane of incidence, so that the incident wave is

$$X = 0, \quad Y = \exp[\iota p\{t - (x\sin\theta + z\cos\theta)/c\}], \quad Z = 0,$$

$$\alpha = -\cos\theta\exp[\iota p\{t - (x\sin\theta + z\cos\theta)/c\}], \quad \beta = 0, \quad \gamma = \sin\theta\exp[\iota p\{\ldots\}].$$

The reflected wave is

$$X = 0, \quad Y = B\exp[\iota p\{t - (x\sin\theta - z\cos\theta)/c\}], \quad Z = 0,$$

$$\alpha = B\cos\theta\exp[\iota p\{t - (x\sin\theta - z\cos\theta)/c\}], \quad \beta = 0, \quad \gamma = B\sin\theta\exp[\iota p\{\ldots\}].$$

Inside the film, *i.e.*, between $z = 0$ and $z = d$,

$$\begin{aligned} X = 0, \quad Y = {} & A'\exp[\iota p\{t - (x\sin\phi + z\cos\phi)/V\}] \\ & + B'\exp[\iota p\{t - (x\sin\phi - z\cos\phi)/V\}], \quad Z = 0. \end{aligned}$$

$$\alpha = -\frac{c\cos\phi}{V}\{A'\exp[\iota p\{t - (x\sin\phi + z\cos\phi)/V\}] - B'\exp[\iota p\{\ldots\}]\}, \quad \beta = 0,$$
$$\gamma = \sin\theta\{A'\exp[\ldots] + B'\exp[\ldots]\}.$$

Transmitted wave

$$X = 0, \quad Y = C\exp[\iota p\{t - (x\sin\theta + z\cos\theta)/c\}], \quad Z = 0,$$

$$\alpha = -C\cos\theta\exp[\iota p\{t - (x\sin\theta + z\cos\theta)/c\}], \quad \beta = 0, \quad \gamma = C\sin\theta\exp[\ldots].$$

In these expressions we have

$$V^2 = c^2/\epsilon \quad \text{and} \quad \sin\phi/V = \sin\theta/c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a).$$

Since Y and α are continuous at $z = 0$,

$$1 + B = A' + B' \quad \text{and} \quad (1 - B)\cos\theta = (A' - B')\,c/V\cos\phi \quad . \quad . \quad (b).$$

Since Y and α are continuous at $z = d$, if we replace

$\exp[\pm\iota p\,d/V\cos\phi]$ by $1 \pm \iota p\,d/V\cos\phi$, and $\exp[-\iota p\,d/c\cos\theta]$ by $1 - \iota p\,d/c\cos\theta$,

we obtain, when the square of $2\pi d/\lambda$ is neglected, the equations

$$\left.\begin{aligned} \{A' + B' - (A' - B')\,\iota p\,d/V\cos\phi\} &= C(1 - \iota p\,d/c\cos\theta) \\ c\cos\phi/V\{(A' - B') - (A' + B')\,\iota p\,d/V\cos\phi\} &= \cos\theta C(1 - \iota p\,d/c\cos\theta) \end{aligned}\right\} \quad . \quad (c).$$

From the last pair of equations we find, neglecting squares of $p\,d/c$, that

$$A' + B' = C$$

$$\frac{c\cos\phi}{V\cos\theta}(A' - B') = C\left\{1 - \frac{\iota p d\sec\theta}{c}\left(\cos^2\theta - \frac{c^2\cos^2\phi}{V^2}\right)\right\} = C\left\{1 - \frac{\iota p d\sec\theta}{c}(1 - \epsilon)\right\} \quad . \quad (d),$$

on using (a); then eliminating A′, B′, B from the equations (b) and (d), we finally have

$$C = 1 - \iota\pi\,d/\lambda\,.\sec\theta\,.(\epsilon - 1).$$

On taking the modulus and substituting for ϵ from (17) we obtain

$$|C|^2 = 1 - 4\pi d/\lambda \sec\theta \,.\, n^2\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Secondly, suppose that the incident light is polarised perpendicular to the plane of incidence, α, β, γ being the magnetic force.

The incident wave is

$$\alpha = 0, \quad \beta = \exp\left[\iota p\left\{t - (x\sin\theta + z\cos\theta)/c\right\}\right], \quad \gamma = 0.$$

The transmitted wave is

$$\alpha = 0, \quad \beta = C\exp\left[\iota p\left\{t - (x\sin\theta + z\cos\theta)/c\right\}\right], \quad \gamma = 0.$$

The velocity V inside the film is connected with the angle of refraction by the equations

$$V^2 = c^2\left(\frac{\cos^2\phi}{\epsilon} + \frac{\sin^2\phi}{\epsilon'}\right) = \frac{c^2\epsilon'}{\epsilon\epsilon' + \sin^2\theta\,(\epsilon' - \epsilon)}.$$

The final result after using the two sets of boundary conditions is

$$C = 1 - \frac{\iota\pi d\sec\theta}{\lambda}\left\{\cos^2\theta\,(\epsilon - 1) + \sin^2\theta\,\frac{\epsilon' - 1}{\epsilon'}\right\},$$

whence, using (17), we obtain

$$|C|^2 = 1 - \frac{4\pi d\cos\theta}{\lambda}\left\{n^2\kappa + \tan^2\theta\,\frac{\epsilon' - 1}{\epsilon'}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23).$$

When the light is normally incident, the crystalline character of the film does not manifest itself, and we have from (22) or (23)

$$|C|^2 = 1 - 4\pi d/\lambda \,.\, n^2\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

The absorption of directly incident light by a thin film is therefore governed by $n^2\kappa$.

Owing to the difficulty of knowing whether any particular metal film whose changes of colour have been observed, but whose thickness has not been recorded, for example, the films observed by Professor R. W. WOOD or by Mr. G. T. BEILBY (*loc. cit.*), is to be regarded as very thin for the purpose of this section, formulæ for thick films will now be found.

We consider here only the case of directly incident plane-polarised light, and proceed to obtain an equation corresponding to (24), reserving the full discussion of the behaviour of thick films under obliquely incident light till later. Using the axes shown in fig. 10, suppose that

Incident wave is

$$\mathbf{E} = 0,\ \exp\{\iota p\,(t - z/c)\},\ 0; \quad \mathbf{H} = -\exp\{\iota p\,(t - z/c\},\ 0,\ 0.$$

Reflected wave

$$\mathbf{E} = 0, \quad B\exp\{\iota p\,(t + z/c)\},\ 0; \quad \mathbf{H} = B\exp\{\iota p\,(t + z/c)\},\ 0,\ 0.$$

Wave in film, *i.e.*, between $z = 0$ and $z = d$,

$$\mathbf{E} = 0, \quad \mathrm{A}' \exp \{\iota p (t - z/\mathrm{V})\} + \mathrm{B}' \exp \{\iota p (t + z/\mathrm{V})\}, \ 0,$$

$$\mathbf{H} = - c/\mathrm{V} [\mathrm{A}' \exp \{\iota p (t - z/\mathrm{V})\} - \mathrm{B}' \exp \{\iota p (t + z/\mathrm{V})\}], \ 0, \ 0.$$

Transmitted wave

$$\mathbf{E} = 0, \quad \mathrm{C} \exp \{\iota p (t - z/c)\}, \ 0; \quad \mathbf{H} = - \mathrm{C} \exp \{\iota p (t - z/c)\}, \ 0, \ 0,$$

where $c/\mathrm{V} = n (1 - \iota\kappa)$.

The boundary conditions at $z = d$ give

$$\left.\begin{aligned} &\mathrm{A}' e^{-2\pi dn\kappa/\lambda} \exp \{- \iota \,.\, 2\pi dn/\lambda\} + \mathrm{B}' e^{2\pi dn\kappa/\lambda} \exp \{\iota \,.\, 2\pi dn/\lambda\} = \mathrm{C} \exp \{- \iota 2\pi d/\lambda\} \\ &n (1 - \iota\kappa) [\mathrm{A}' e^{-2\pi dn\kappa/\lambda} \exp \{- \iota \,.\, 2\pi dn/\lambda\} - \mathrm{B}' e^{2\pi dn\kappa/\lambda} \exp \{\iota 2\pi dn/\lambda\}] \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \mathrm{C} \exp \{\iota 2\pi d/\lambda\} \end{aligned}\right\} \quad (25).$$

It follows from these equations that B' is of the order of $\mathrm{A}' e^{-4\pi dn\kappa/\lambda}$; if therefore $\pi dn\kappa/\lambda > 1$ we shall be correct within 2 per cent. (e^{-4}) when we neglect B'. Thus referring to the Table IV. it appears that if a piece of gold leaf before annealing be so thick that $d > \lambda/1{\cdot}5$ or $d > \frac{2}{3}\lambda$, then, so far as yellow and red light are concerned, $\pi dn\kappa/\lambda$ will be > 1 for all values of $\mu \geqq {\cdot}5$, if we suppose d to vary inversely as μ, the number $1{\cdot}5$ being the smallest value of $\pi n\kappa/\mu$ for gold for values of μ from ·5 up to unity.

Eliminating $\mathrm{B}' e^{2\pi dn\kappa/\lambda}$ from the last two equations above,

$$2\mathrm{A}' e^{-2\pi dn\kappa/\lambda} = \mathrm{C} \left\{ \frac{1 + n (1 - \iota\kappa)}{n (1 - \iota\kappa)} \right\} \exp \{- \iota 2\pi d/\lambda (1 - n)\}.$$

From the boundary condition at $z = 0$ we obtain

$$\mathrm{A}' \{1 + n (1 - \iota\kappa)\} = 2.$$

Eliminating A' from the last two equations

$$\mathrm{C} \exp \{- \iota 2\pi d/\lambda (1 - n)\} = 4 e^{-2\pi dn\kappa/\lambda} \left[\frac{n (1 - \iota\kappa)}{\{1 + n (1 - \iota\kappa)\}^2} \right].$$

Taking the moduli, the ratio $|\mathrm{C}|^2$ of the intensity of the transmitted to that of the incident light is given by

$$|\mathrm{C}|^2 = \frac{16 n^2 (1 + \kappa^2)}{\{(1 + n)^2 + n^2\kappa^2\}^2} e^{-4\pi dn\kappa/\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

It appears that when the thickness exceeds $\frac{2}{3}$ of the wave-length, the absorption is governed by $n\kappa$; but, to the same order of approximation, by $n^2\kappa$ when d is less than $\frac{1}{25}\lambda$.

The (comparatively) small effect of the coefficient

$$M_0 \equiv \frac{16n^2(1+\kappa^2)}{\{(1+n)^2 + n^2\kappa^2\}^2}$$

on the colour by transmitted light will be considered later. For the present it is sufficient to observe that M_0 becomes small when $\mu = 1/\alpha$ for any colour, and hence that the variations of M_0 intensify the absorption bands, which will be shown to occur for gold and silver near $\mu\alpha = 1$.

§ 10. In order to illustrate the discussion of the colours exhibited by films of metal for various values of μ, graphs are given of $n\kappa$ and $n^2\kappa$ for gold, for silver, and for the amalgam, potassium-sodium, the constants of which for $\mu = 1$ were given by DRUDE (*loc. cit.*). The graphs of $n\kappa$ and $n^2\kappa$ for gold and for silver when the incident light is red or yellow are plotted from the values given in the accompanying table, with the help of the additional point corresponding to $\mu\alpha = 1$.

This last point is easy to plot, for we see from equations (18) and (19) that $n^2\kappa = \frac{3}{4\mu\beta} = \frac{3\alpha}{4\beta}$ and $n^2(\kappa^2 - 1) = 2$ when $\mu = 1/\alpha$.

Consequently for

$$\mu = 1/\alpha,\ n^2(\kappa^2+1) = 2\sqrt{n^4\kappa^2 + 1},$$

so that

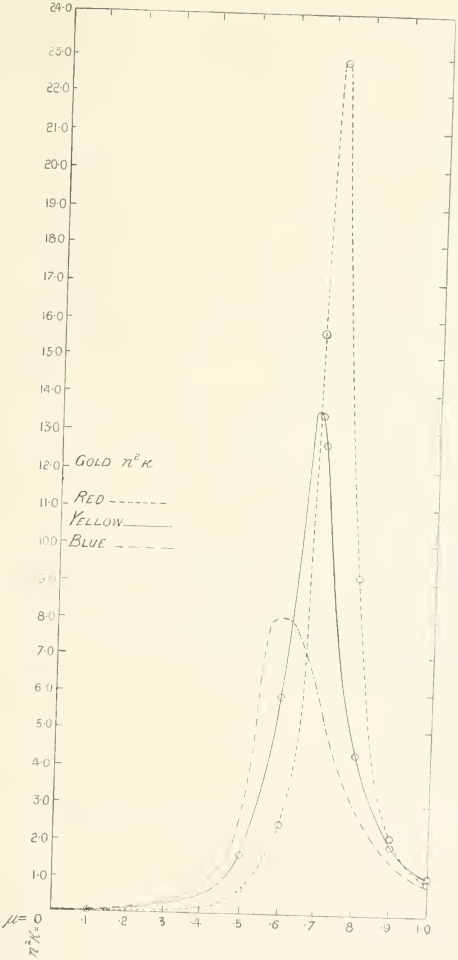
$$n^2\kappa^2 = \sqrt{n^4\kappa^2 + 1} + 1 \quad \text{and} \quad n\kappa = \left\{\sqrt{\left(\frac{3\alpha}{4\beta}\right) + 1} + 1\right\}^{\frac{1}{2}}.$$

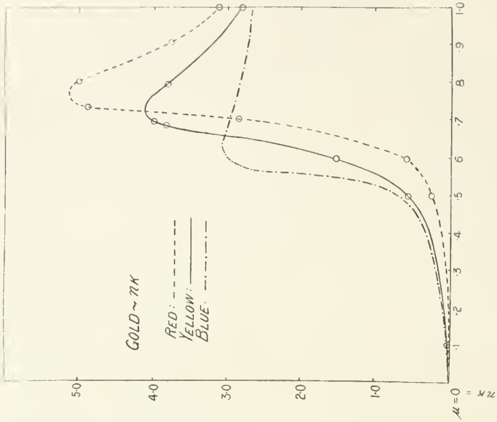
This point is also very near the maximum of $n^2\kappa$, owing to the smallness of β in comparison with α, and is also, in the graph of $n\kappa$, not far from the maximum, and in the graph of M_0 not far from the minimum. It will be shown that for each of these reasons there is in general an absorption band in the colour whose $\alpha = 1/\mu$.

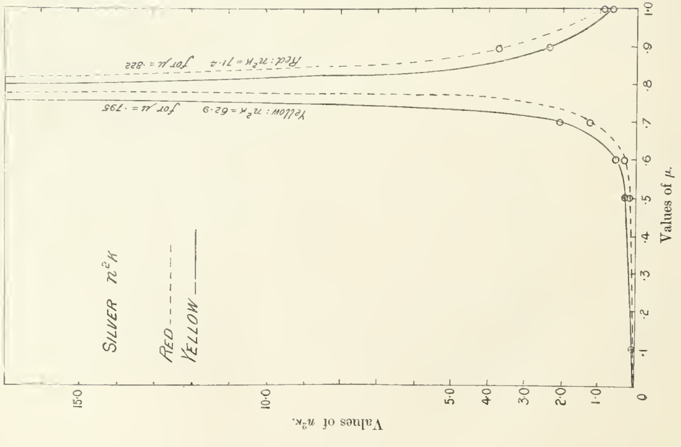
The graph for $n^2\kappa$ when blue light is incident on gold is surmised; *i.e.*, it is constructed on the supposition that the constants n and κ for gold, when $\mu = 1$, are continuous from red through yellow to blue. The curve for $n\kappa$ for gold under blue light is made of the same shape as for yellow and red, the value of $n\kappa$ for $\mu = \alpha^{-1}$ being plotted from the maximum value of $n^2\kappa$ assumed in that graph.

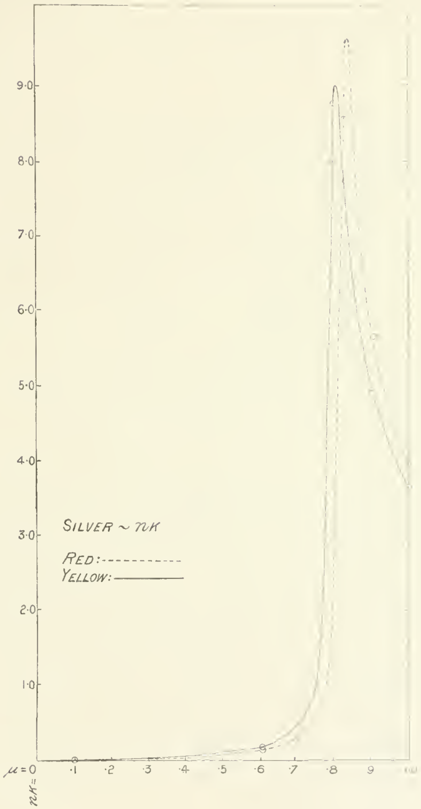
The curves for $n^2\kappa$ for potassium-sodium are plotted, again with the help of the points for $\mu = 1/\alpha$, the incident light being blue or yellow. The graph of $n^2\kappa$ for red is again surmised.

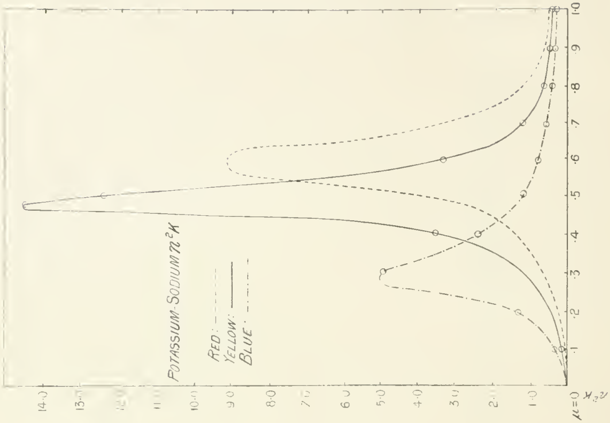
The graphs of $n\kappa$ for potassium-sodium are shown by analogy with those for gold, the only points plotted being for $\mu = 1/\alpha$, $\mu = 1$. The red curve is constructed from that for $n^2\kappa$ in the same manner as the blue curve for $n\kappa$ for gold was got from the assumed curve for $n^2\kappa$ for gold.

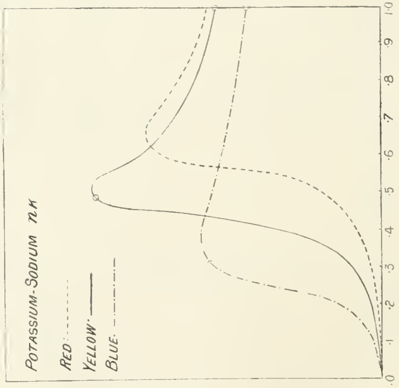

Gold $n^2\kappa$
Red
Yellow
Blue
24·0
23·0
22·0
21·0
20·0
19·0
18·0
17·0
16·0
15·0
14·0
13·0
12·0
11·0
10·0
9·0
8·0
7·0
6·0
5·0
4·0
3·0
2·0
1·0
μ= 0
$n^2\kappa$=
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0


GOLD ~ nκ
RED: - - - -
YELLOW: ——
BLUE: —·—·—
5·0
4·0
3·0
2·0
1·0
μ = 0 = nκ
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0


SILVER n²κ
RED - - - -
YELLOW ——
Red: n²κ = 71·4 for μ = ·822
Yellow: n²κ = 62·9 for μ = ·795
15·0
10·0
5·0
4·0
3·0
2·0
1·0
0
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0
Values of μ.
Values of n²κ.


SILVER ~ nκ
RED:-----------
YELLOW:———
9·0
8·0
7·0
6·0
5·0
4·0
3·0
2·0
1·0
μ = 0
nκ =
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0


POTASSIUM-SODIUM n^2K
RED:
YELLOW:
BLUE:
14·0
13·0
12·0
11·0
10·0
9·0
8·0
7·0
6·0
5·0
4·0
3·0
2·0
1·0
$\mu=0$
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0


POTASSIUM-SODIUM nK
RED:
YELLOW:
BLUE:
·0
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0

§ 11. In a paper on "The Effects of Heat and of Solvents on Thin Films of Metal," 'Roy. Soc. Proc.,' vol. 72, 1903, p. 226, Mr. G. T. BEILBY gives an account of some experiments on the behaviour of gold and silver films when heated to temperatures far below their melting points. He suggests that at such temperatures sufficient freedom is conferred on the molecules by the heating to enable them to behave as the molecules of the liquid metal would do, and to arrange themselves under the influence of surface tension either in films or in drop-like granular forms.

We have already shown, when dealing with the colours in metal glasses, how the small particles of metal excrete themselves from the glass into spherical forms.

Mr. BEILBY records that the resistance of silver and gold films increased, on annealing, from a few (0·2 up to 50) ohms up to many thousands of megohms. This, of course, strongly supports the theory that the metal breaks up under surface tension into minute granules. Professor WOOD observed no conductivity in his films as originally deposited. Mr. BEILBY further states that in one of the gold films there appeared to be a considerable depth of granules, and Professor WOOD records absence of conductivity in a film in which granules appeared in contact with and piled upon top of one another. These observations support our hypothesis as to the structure of the films, although the granules observed may have been larger than those which are effective in producing the colour phenomena which we are to investigate.

Let us now see whether our hypothesis as to the structure of the films is in agreement with the colour effects observed by Mr. BEILBY and Professor WOOD.

First, then, consider a very thin film of gold.

According to the result given in equation (24) the diminution in intensity of light of wave-length λ is, for such a film, $4\pi d/\lambda \,.\, n^2\kappa$.

From the graphs of $n^2\kappa$ for yellow and for red, it is seen that for the solid metal for which $\mu = 1$, *i.e.*, before annealing, $n^2\kappa/\lambda$ is less for red than for yellow, and this is true from $\mu = 1$ nearly down to $\mu = \cdot 9$. Thus, a very thin leaf of gold should *not* show the green colour distinctive of gold leaf, but the red colour should predominate over the yellow. The arbitrary graph for $n^2\kappa$ for blue would, if correct, show that blue should predominate over either yellow or red.

The colour of a very thin film of gold leaf would, therefore, be chiefly blue, less red, and least yellow, *i.e.*, blue-purple, and this is the colour observed by Mr. BEILBY in the thinnest piece of gold leaf he possessed (*loc. cit.*, p. 227).

It should be noticed that it has been *proved* that a very thin film will let through more red than yellow light, and that it, therefore, will not exhibit the green colour of gold leaf. It has only been stated that it seems *probable* that it will let through more blue than either.

We now suppose that when the film is being annealed, surface tension acts and causes the gold to form into spherical drops, many to a wave-length, but of quite varying sizes. Thus μ, the volume of metal in a unit volume of the film, continuously diminishes from unity downwards so long as the metal is kept at a temperature of about 400°.

Just before μ has reached ·9 the yellow begins to get through better than the red, but the absorption of both increases rapidly. The value of $n^2\kappa$ for red becomes equal to 24 when $\mu = \cdot734$ about. It almost immediately starts to diminish, being only 15·88 when $\mu = \cdot7$. There is thus a strong and quite narrow absorption band in the red for $\mu = \cdot734$.

Similarly, $n^2\kappa$, when the incident light is yellow, rises to a high value near $\mu = \cdot686$, and when μ has that value, $n^2\kappa = 15$ nearly.

Between $\mu = \cdot7$ and $\mu = \cdot734$, red and yellow are absorbed to the same large extent. It seems probable that blue will not be absorbed so greatly for this value of μ. The film should therefore probably be blue. Mr. BEILBY finds (*loc. cit.*, p. 228) that a gold film turns blue or purple (absorption chiefly of yellow) in the earlier stage of annealing, though, presumably, the films for which this effect was observed fall into our class of thick films. The turning blue will therefore be again referred to when we come to consider thick films of gold.

When $\mu = \cdot6$, the red light is much less diminished in intensity than the yellow, and probably less than the blue. The film is therefore pink, and remains pink down to the dimensions of coloured glass for which μ is of the order of 10^{-5}. The thin film observed by Mr. BEILBY was rose pink after annealing (p. 227).

The high transparency observed by FARADAY and by Mr. BEILBY corresponds to the very small values of $n^2\kappa$ for values of $\mu < \cdot5$.

Consider next a thick film of gold.

The absorption being now, according to the result given on p. 409, dependent principally upon the value of $n\kappa/\lambda$, we see from the table for $n\kappa$ or from the graph that, for the solid metal, yellow light is less absorbed than red. The colour of thick gold leaf is, in fact, olive-green by transmitted light. As μ diminishes the absorption of both yellow and red increases, the latter more rapidly. Now when $\mu = \cdot734$, there is a great absorption of red, according to the values of $n\kappa$, which is intensified, since M_0 is for this value of μ reduced to ·177. The colour should then be more yellow than red, and probably more blue than either. When μ is $< \cdot7$, the colour is much more red than yellow. If our assumed curve for $n\kappa$ for blue is correct, the colour of the film should be blue between $\mu = \cdot85$ and ·7, purple at ·7, and principally red from $\mu = \cdot65$ through all the range of values of μ from gold glass down to $\mu = 0$. (If our curve for blue is correct, the figure shows that the film is red when the blue curve crosses the yellow.)

According to Mr. BEILBY, a gold film, originally green, turned blue-purple after annealing. Gold leaf turned, by annealing, pink with brown-green patches, the latter, presumably, corresponding to large and the former to small values of μ.

The rise in the absorption as μ begins to diminish from unity was noticed by Mr. BEILBY (*loc. cit.*, p. 232).

It has therefore been shown that all the observed colour changes in gold films are in accordance with the theory and numerical results set forth in this paper.

The points corresponding to $\mu\alpha = 1$, referred to on p. 410, which were plotted in for red and yellow, were

$$n\kappa = 4{\cdot}89 \text{ for } \mu = {\cdot}734 \text{ for red,}$$
$$n\kappa = 3{\cdot}80 \text{ ,, } \mu = {\cdot}685 \text{ ,, yellow.}$$

Let us now consider silver films.

The results for *thin* films are not of much interest, as probably none of the films observed came into this class. We may, however, notice that, according to the graphs of $n^2\kappa$, the thin film should start by being more yellow than red. There is an absorption band in the red about $\mu = {\cdot}822$, for which value of μ, $n^2\kappa = 71{\cdot}4$ for red. There is great absorption in the yellow for $\mu = {\cdot}795$ when $n^2\kappa = 62{\cdot}9$. The nearness of these values of μ for the maxima of the absorption of yellow and red suggests that thin films of silver should be blue or else very opaque when μ is about ${\cdot}8$. The thin film should turn more red than yellow for μ slightly $> {\cdot}8$ and remain red down to small values of μ, at least as far as $\mu = {\cdot}1$.

Passing to thick films, for which the absorption is measured by $n\kappa\lambda$, we observe from the graphs that as μ diminishes from unity the absorption at first increases rapidly. This may be correlated with the increased conductivity manifested by a silver film in the early stages of annealing. Shortly before $\mu = {\cdot}8$ the film becomes more red than yellow, and although by the time $\mu = {\cdot}6$ the absorption has already become extremely small, the film remains more red than yellow until μ vanishes.

Putting $\mu = 1/\alpha$, we find the additional points on the graphs of $n\kappa$

for red . . . . $n\kappa = 8{\cdot}51$ when $\mu = {\cdot}822$.
,, yellow . . . $n\kappa = 7{\cdot}99$,, $\mu = {\cdot}795$.

The red colour of silver films for low values of μ is observed in those obtained by depositing silver on glass in the manner described by Professor Wood ('Phil. Mag.,' August, 1903). It is also often seen in fogged photographic plates.

§ 12. We proceed to consider the potassium and sodium films discussed by Professor R. W. Wood, in the 'Phil. Mag.,' 1902, p. 396, *et seq.* Owing, however, to the inavailability of the numerical values of the constants for potassium or sodium for more than one colour when $\mu = 1$, the numbers used are those given by Drude (*loc. cit.*) for "potassium-sodium," for blue and yellow light. Consequently the same degree of numerical accuracy as for gold and silver has not been aimed at.

The yellow and blue curves for $n^2\kappa$ are plotted from the numbers tabulated in Table IV., p. 406.

The graph of $n\kappa$ has been constructed to pass through the untabulated points

Yellow . . . $n\kappa = 3{\cdot}811$ for $\mu = 1/\alpha = {\cdot}484$, $n\kappa = 2{\cdot}18$ for $\mu = 1$.
Blue . . . . $n\kappa = 2{\cdot}225$,, $\mu = 1/\alpha = {\cdot}306$, $n\kappa = 1{\cdot}78$,, $\mu = 1$.

The films made by Professor WOOD were obtained by the condensation of the vapour of the evaporated metal on the insides of exhausted glass bulbs. We should therefore expect the film in its original form to be in drops, which, in accordance with Part I., § 3, when very small, approximate to the spherical form.

The absence of conductivity in these films supports this view of their structure. The effect of heating up to and beyond the melting point would be to fuse these drops into continuous metal, and although surface tension tends to a re-formation into spheres, it is probable that μ will generally be considerably increased by the fusion.

It appears, from our graphs of $n^2\kappa$ and $n\kappa$, that thin or thick "potassium-sodium" films should transmit more blue than yellow light, provided that $\mu \geqq \cdot 4$, there being a very strong absorption of yellow for $\mu = \cdot 49$ (about) in both cases. It is interesting to note that Professor WOOD always refers to the yellow absorption bands as particularly strong. As μ increases, the absorption of yellow relative to blue increases in both thick and thin films.

If now we introduce our hypothetical curves for $n^2\kappa$ and for $n\kappa$ for red light, we find that for $\mu \leqq \cdot 4$ the film should be red. Near the greatest absorption of yellow ($\mu = \cdot 49$), red and blue should be equally absorbed and the film be purple. As μ increases further, red should be most absorbed, and blue least, so the film should be blue. Thus, in general, the film should turn from purple to blue when heated, as was the case with most of the films observed. Professor WOOD (*loc. cit.*, p. 407) further states that the particles which he observed were distinctly closer in the blue than in the pinkish-purple part, thus again suggesting that a change from purple to blue accompanies an increase in μ.

So far, then, as they go, our results are in good accordance with observation. When, however, numbers can be obtained for n and $n\kappa$ for potassium and for sodium for blue, yellow, and red light, it may be possible to state with more certainty that our explanation of the colours of the films and of the changes in colour due to heating is the true one.

§ 13. By considering the oblique incidence of plane-polarised light on thick films of metal by the method adopted in § 9 in the case of thin films, it can be shown that equation (26) is replaced by :—

(1.) When the incident light is polarised in the plane of incidence

$$|C'|^2 = \frac{16\,(u^2 + v^2)}{\{(1 + u)^2 + v^2\}^2}\, e^{-4\pi d\kappa/\lambda\,.\,\cos\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

(2.) When the incident light is polarised perpendicular to the plane of incidence

$$|C'|^2 = \frac{16\,(u'^2 + v'^2)}{\{(1 + u')^2 + v'^2\}^2}\, e^{-4\pi d\kappa/\lambda\,.\,\cos\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

where u, v and u', v' are certain functions of n, κ, θ such that when the angle of incidence, θ, is zero,

$$u = u' = n \quad \text{and} \quad v = v' = n\kappa.$$

It can further be proved that the variations with μ of the coefficients

$$M_\theta \equiv \frac{16\,(u^2 + v^2)}{\{(1 + u)^2 + v^2\}^2} \quad \text{and} \quad M'_\theta \equiv \frac{16\,(u'^2 + v'^2)}{\{(1 + u')^2 + v'^2\}^2}$$

are such that a change in (27) from M_θ to M_θ' would strengthen the absorption bands. The complete analysis is somewhat lengthy; I have therefore refrained from reproducing it here.

This result, however, shows that in general the absorption band should be weaker when the incident light is polarised in the plane of incidence than when it is polarised perpendicular to that plane. And this effect Professor WOOD observed in almost every film.

PART III.

§ 14. Metallic media composed of small spheres of metal, many to a wave-length, have many interesting properties in addition to those already referred to. The very vivid colour effects which are exhibited according to the graphs given above for $n\kappa$ for gold, silver and "potassium-sodium" when light traverses such media, in consequence of the different absorptions of different colours, suggest enquiry whether metals in bulk have ever been obtained giving brilliant colours by transmitted and reflected light, such metals being ordinary metals with μ less than unity. For instance, have any of the metals we have discussed been obtained in states in which the specific gravity was not the normal value for that metal and in which the colour changed with the specific gravity?

I hope in the near future to examine CAREY LEA's work in detail with a view to finding out whether his allotropic silver is a medium of the type we have considered—silver with μ less than unity. But the first glance at his papers ('American Journal of Science,' 1889) shows the following remarkable correspondence between the properties he observed and the properties which should, according to our calculations for yellow and red light, be possessed by silver with $\mu < 1$:—

(i.) CAREY LEA's silvers were obtained from solution; and we have shown that gold, and therefore, presumably, silver, crystallises out of solutions into particles which are spherical if they are very small. Our silver ($\mu < 1$) is composed of minute spheres.

(ii.) CAREY LEA's silver can be changed by pressure or heating into normal silver. We should expect μ to be increased by pressure.

(iii.) The specific gravities of the two principal forms of allotropic silver were appreciably less than that of normal silver.

(iv.) From our graph of $n\kappa$ for silver we see that red and yellow light are about equally, and very powerfully, absorbed when $\mu = \cdot 81$. The ratio of the specific gravities of CAREY LEA's gold-coloured silver, C, and normal silver is given by him to be 8·51/10·62 = ·81. This strongly supports the theory that allotropic silver is of the nature of the media we have discussed.

(v.) CAREY LEA's silvers were very brittle, but could be toughened by heating. Further, his gold-coloured silver could be transformed into normal silver by shaking; and this transformation could be greatly impeded by packing the gold-coloured silver in cotton wool. These properties suggest a discontinuous structure for allotropic silver.

(vi.) If we might assume an absorption graph of $n\kappa$ for blue light, the fact that if light is obliquely reflected from a film of "B" silver, then the yellow light is polarised in the plane of incidence and the blue perpendicular to that plane can, I think, be explained by our theory: but the proof is not yet complete.

(vii.) The red colour exhibited by all the more dilute forms of the allotropic silver is in accordance with the fact, exhibited by the graph, that $n\kappa$ is smaller for red than for yellow light for small values of μ.

[APPENDIX, *added 28th July*, 1904.—Using the values of the refractive index and absorption coefficient of gold for red (C), green (E), and blue light, as given by RUBENS ('Wied. Ann.,' 1889), the following values of the quantity β/λ, which governs the absorption of the gold glass, have been calculated:—

		Red (C).	Red (·630).	Yellow (D).	Green (E).	Blue $\frac{1}{2}$(F + G).
Gold	n	·38	·31	·37	·53	·79
	$n\kappa$	2·91	3·15	2·82	1·86	1·52
	β	·48	·25	·59	1·07	·46
	β/λ	·73	·40	·99	2·03	1·01

The refractive index of the glass has been taken to be 1·56, as in Table II., from which the values of β for red and for yellow have been copied.

The colours, in the order of the degree in which they are transmitted by gold glass, therefore are

Red, Yellow, Blue, Green.

The corresponding order for silver as obtained by calculation is

Yellow, Red, Green, Blue.

The orders accord with observations on gold-ruby glasses and silver glasses respectively.

It will be seen that large particles of gold (diameter > 0·1 μ) in a gold glass would, by reflecting out the red and yellow light, give the glass a blue colour by transmitted light, and a brown turbidity by reflected light—as in glasses D of Table I.]

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